$$v_p v_g = c^2$$

Consider a particle:

$$E^2 = p^2 c^2 + m_0^2 c^4 : E = \hbar\omega : p = \hbar k$$

Then:

$$\hbar^2\omega^2 = \hbar^2 k^2 c^2 + m_0^2 c^4$$

$$\omega^2 = k^2 c^2 + \frac{m_0^2 c^4}{\hbar^2}$$

differentiate with respect to k:

$$2\omega\frac{d\omega}{dk} = 2kc^2$$

$$\Rightarrow \frac{d\omega}{dk} = \frac{kc^2}{\omega}$$

But:

$$\frac{d\omega}{dk} = v_g \text{ and } \frac{\omega}{k} = v_p$$

We thus have:

$$v_p v_g = c^2$$

In a medium characterized by n(ω) we have :

$$v_g = v_p + k\frac{dv_p}{dk}$$

but in a material we have:

$$\frac{c}{v_p} = n(\omega) \text{ so } v_g = \frac{c}{n + \omega\frac{dn}{d\omega}}$$

note that if dn/dω is negative and sufficiently large, v_g can exceed c but the energy propagation is governed by the front velocity which is bounded by c.