8.2.2 The Maxwell Stress Tensor

2020

Consider charges within a volume that may be moving and responding to forces.

The charges would exert, and respond to the Lorentz forces:

$$\vec{F}=q\vec{E}+q\vec{v}\times\vec{B}$$

If we have a volume charge distribution, we can do this:

$$\vec{F}=\oiiint_{volume}\left(\rho\vec{E}+\rho\vec{v}\times\vec{B}\right)d^3r=\oiiint_{volume}\left(\rho\vec{E}+\vec{J}\times\vec{B}\right)d^3r$$

The force per unit volume would then be:

$$\vec{f}=\rho\vec{E}+\vec{J}\times\vec{B}$$

This is then what we are going to work with, and eliminate the explicit reference to charges and currents (we're assuming conventional currents here)

$$\text{Gauss's law}:\rho=\epsilon_0\vec{\nabla}\cdot\vec{E}$$

$$\text{Ampere's law}:\vec{J}=\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}-\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

Then the force per unit volume becomes:

$$\vec{f}=\epsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E}+\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

Let's look at the last term: (and using Faraday's law)

$$\epsilon_0\frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}=\epsilon_0\left[\frac{\partial\vec{E}}{\partial t}\times\vec{B}+\vec{E}\times\frac{\partial\vec{B}}{\partial t}\right]=\epsilon_0\left[\frac{\partial\vec{E}}{\partial t}\times\vec{B}-\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\right]$$

So: the last term is:

$$\epsilon_0\frac{\partial\vec{E}}{\partial t}\times\vec{B}=\epsilon_0\frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}+\epsilon_0\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)$$

This let's us write the force per unit volume as the ugly expression:

$$\vec{f}=\epsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E}+\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\epsilon_0\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)-\epsilon_0\frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}$$

$$\vec{f}=\epsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E}+\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\epsilon_0\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)-\epsilon_0\frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}$$

Let's work with these terms separately:

First: $\epsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E}-\epsilon_0\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)$:

$$\vec{\nabla}\left(E^2\right)=2\left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}+2\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)\Rightarrow\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)=-\left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}+\frac{\vec{1}}{2}\nabla\left(E^2\right)$$

Now $\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}=\left(\vec{\nabla}\cdot\vec{B}\right)\vec{B}+\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}$ Note addition of zero!

This allows us to write the force per unit volume which was

$$\vec{f}=\epsilon_0\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E}+\frac{1}{\mu_0}\left(\vec{\nabla}\times\vec{B}\right)\times\vec{B}-\epsilon_0\vec{E}\times\left(\vec{\nabla}\times\vec{E}\right)-\epsilon_0\frac{\partial\left(\vec{E}\times\vec{B}\right)}{\partial t}$$

as

$$\vec{f}=\epsilon_0\left[\left(\vec{\nabla}\cdot\vec{E}\right)\vec{E}+\left(\vec{E}\cdot\vec{\nabla}\right)\vec{E}\right]+\frac{1}{\mu_0}\left[\left(\vec{\nabla}\cdot\vec{B}\right)\vec{B}+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{B}\right]-\frac{1}{2}\vec{\nabla}\left(\epsilon_0E^2+\frac{1}{\mu_0}B^2\right)-\epsilon_0\frac{\partial}{\partial t}\left(\vec{E}\times\vec{B}\right)$$

Now introducing the Maxwell Stress Tensor with elements given by:

$$T_{ij} \equiv \epsilon_0 \left(E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right) + \frac{1}{\mu_0} \left(B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right)$$

Where i,j refer to x,y,z and the Kronecker delta is given by

$$\delta_{ij} = \begin{Bmatrix} 1, i=j \\ 0, i \neq j \end{Bmatrix}$$

$\overset{\Leftrightarrow}{T}$ indicates a tensor, (till I find out a better way to make it with openoffice) containing all (9) components and the operation is:

$$(\vec{A} \cdot \overset{\Leftrightarrow}{T}) = \sum_{i=x,y,z} A_i T_{ij} \quad \hat{i}$$

and the vector a can be a differential operator.

Remember that: the Poynting vector is $\vec{S} \equiv \frac{1}{\mu_0} (\vec{E} \times \vec{B})$

With this we can then write the force per unit volume as:

$$\vec{f} = \vec{\nabla} \cdot \overset{\Leftrightarrow}{T} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}$$

We can then find the total force from :

$$\vec{F} = \oiint \overset{\Leftrightarrow}{T} \cdot d\vec{A} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} \oiiint \vec{S} \, d^3 r$$

I recommend looking at Example 8.2 since it is a good one but we will pass over it.

8.2.3 Momentum

Now that we know about the Maxwell Stress Tensor, we can use it to look at conservation of momentum in an EM field.

From Newton's second law, we know:

$$\vec{F} = \frac{d\vec{P}_{mech}}{dt} \Rightarrow \frac{d\vec{P}_{mech}}{dt} = \oiint \overset{\Leftrightarrow}{T} \cdot d\vec{A} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} \oiiint \vec{S} \, d^3 r$$

which is the statement of the conservation of momentum including EM fields.

The momentum stored in the EM field and the momentum density is given by:

$$\vec{P}_{EM} = \epsilon_0 \mu_0 \oiiint \vec{S} \, d^3 r \Rightarrow \vec{p}_{EM} = \epsilon_0 \mu_0 \vec{S}$$

With this, we have:

$$\frac{\partial}{\partial t}\left[\vec{p}_{mech}+\vec{p}_{EM}\right]=\vec{\nabla}\cdot\overleftrightarrow{T}$$

I have used the divergence theorem to express as a volume integral which was then removed because we are talking about densities.

T expresses the momentum flux density : your author compares it to a current or an energy flux density.

An example is exactly what everyone needs:
Example 8.3: Coaxial cable: length h, inner conductor radius a, +λ, outer conductor radius b, -λ, battery connected between inner and outer conductors, at the other end is a resistor. Inner conductor current flows I in +z, outer conductor is I in-z. Resistor is on +z end. Then, calculate the momentum stored in the EM field.

$$\vec{E}=\frac{\lambda}{2\pi\epsilon_0 s}\hat{s}:\vec{B}=\frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$\vec{S}\equiv\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)=\frac{\lambda I}{4\pi^2\epsilon_0 s^2}\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\end{bmatrix}=\frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{z}$$

The power transported is given by:

$$P_{ower}=\oiint\vec{S}\cdot d\vec{A}=\frac{\lambda I}{2\pi\epsilon_0}\int_{s=a}^{s=b}\frac{ds}{s}=\frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)=IV$$

Now to calculate the momentum:

$$\vec{P}_{EM}=\epsilon_0\mu_0\oiiint\vec{S}\,d^3r=\frac{\mu_i\lambda I h}{2\pi\epsilon_0}z\ln\left(\frac{b}{a}\right)\hat{z}$$

Note your author's discussion on page 357: since the wire is stationary there must be a source of mechanical momentum which is not accounted for; this is contained within the moving charges in this case and it exactly cancels the momentum contained within the fields. See chapter 12.

Right now, we will not pursue this by increasing R , thus decreasing B and inducing an emf which in turn can be shown to exert an impulse on the wire giving a change in momentum equal and opposite to the momentum in the EM field. It has to do with a rather odd constant K which is introduced.

8.2.4 Angular Momentum

Now that we have found the momentum density to be given by:

$$\vec{p}_{EM} = \epsilon_0 \mu_0 \vec{S} = \epsilon_0 \left(\vec{E} \times \vec{B} \right)$$

we can find the angular momentum density which is given by:

$$\text{angular momentum} = \vec{l} = \vec{r} \times \vec{p}_{EM} = \epsilon_0 \left[\vec{r} \times \left(\vec{E} \times \vec{B} \right) \right]$$

The interpretation is this: static E and B fields contain translational and angular momentum and this must be considered in the classical expressions if the total angular and translational momentum is to be conserved.

Note example 8.4: it is indeed an excellent example however we are not going to have time to pursue it this semester.