More applications of Laplace’s Equation to 2 dimensions

Consider a pipe with the following bc:
bc: v(0,y)=v1:v(b,y)=v2:v(x,0)=0:v(x,a)=0

Approach:
Solve this first:
bc: v(0,y)=v1:v(b,y)=0:v(x,0)=0:v(x,a)=0

Then solve this:
bc: v(0,y)=0:v(b,y)=v2:v(x,0)=0:v(x,a)=0

Solution is superposition of the two solutions.

$$\vec{\nabla}^2 V = 0 \Rightarrow \tfrac{1}{X}\tfrac{\partial^2 X}{\partial x^2} + \tfrac{1}{Y}\tfrac{\partial^2 Y}{\partial y^2} = 0 \Rightarrow V = \left(Ae^{-kx} + Be^{+kx}\right)\left(C\sin(ky) + D\cos(ky)\right)$$

BC: v(x,0)=0 so D=0

$$V = \left(Ae^{-kx} + Be^{+kx}\right)\sin(ky)$$

BC: v(x,a)=0 so

$$ka = n\pi \Rightarrow V = \left(Ae^{-\frac{n\pi x}{a}} + Be^{\frac{n\pi x}{a}}\right)\sin\left(\tfrac{n\pi y}{a}\right)$$

Problem now branches:
Apply bc at x=b:

$$V = \left(Ae^{-\frac{n\pi b}{a}} + Be^{\frac{n\pi b}{a}}\right)\sin\left(\tfrac{n\pi y}{a}\right) = 0 \Rightarrow \left(Ae^{-\frac{n\pi b}{a}} + Be^{\frac{n\pi b}{a}}\right) \Rightarrow A = -Be^{\frac{2n\pi b}{a}}$$

Solution is now:

$$V = B'\left(-e^{\frac{2n\pi b}{a}}e^{-kx} + e^{+kx}\right)\sin(ky) = B\left(e^{\frac{n\pi}{a}(x-b)} - e^{-\frac{n\pi}{a}(x-b)}\right)\sin\left(\tfrac{n\pi}{a}y\right)$$

We can write this in terms of the sinh:

$$V_n = B_n \sinh\left(\tfrac{n\pi}{a}(x-b)\right)\sin\left(\tfrac{n\pi}{a}y\right)$$

The general solution is then given by:

$$V(x,y) = \sum_{n=1}^{\infty} V_n = \sum_{n=1}^{\infty} B_n \sinh\left(\tfrac{n\pi}{a}(x-b)\right)\sin\left(\tfrac{n\pi}{a}y\right)$$

We can do the Fourier analysis as before:
at x=0, we have:

$$V_1 = V(0,y) = \sum_{n=1}^{\infty} B_n \sinh\left(-\tfrac{n\pi b}{a}\right)\sin\left(\tfrac{n\pi}{a}y\right)$$

$$\int_0^a V_1 \sin\left(\tfrac{m\pi y}{a}\right)dy = \sum_{n=1}^{\infty} B_n \sinh\left(-\tfrac{n\pi b}{a}\right)\int_0^a \sin\left(\tfrac{n\pi}{a}y\right)\sin\left(\tfrac{m\pi}{a}y\right)dy = \tfrac{a}{2}B_m \sinh\left(-\tfrac{m\pi b}{a}\right)$$

$$\int_0^a V_1 \sin\left(\tfrac{m\pi y}{a}\right)dy = \begin{bmatrix} \frac{2V_1 a}{m\pi} \text{ if m is odd} \\ 0 \text{ if m is even} \end{bmatrix}$$

This gives us:

$$\frac{a}{2}B_m \sinh\left(-\frac{m\pi b}{a}\right)=\frac{2V_1a}{m\pi}\Rightarrow B_m=\frac{4V_1}{m\pi}\frac{1}{\sinh\left(-\frac{m\pi b}{a}\right)}$$

This is thus the general solution:

$$V_{\#1}(x,y)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4V_1}{m\pi}\frac{\sinh\left(\frac{n\pi}{a}(x-b)\right)}{\sinh\left(-\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi}{a}y\right)$$

now we need to do the second branch:

$$V_n(x,y)=\left(Ae^{-\frac{n\pi x}{a}}+Be^{\frac{n\pi x}{a}}\right)\sin\left(\frac{n\pi y}{a}\right)$$

at x=0, V=0 so we thus have:

A=-B. The two solutions then add up to be a sinh again:

$$V_n(x,y)=B_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

At x=b, we thus have:

$$V_2=\sum_{n=1}^{\infty}B_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right)$$

$$\int_0^a V_2 \sin\left(\frac{m\pi y}{a}\right)dy=\sum_{n=1}^{\infty}B_n \sinh\left(\frac{n\pi b}{a}\right)\int_0^a \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy=B_m \sinh\left(\frac{n\pi b}{a}\right)\frac{a}{2}$$

We can thus write the second solution:

$$V_{\#2}=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4V_2}{n\pi}\frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\sin\left(\frac{n\pi y}{a}\right)$$

I can now write the general solution as:

$$V(x,y)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\left[V_1\frac{\sinh\left(\frac{n\pi}{a}(x-b)\right)}{\sinh\left(-\frac{n\pi b}{a}\right)}+V_2\frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\right]\sin\left(\frac{n\pi}{a}y\right)$$

This can be simplified a bit:

$$V(x,y)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi\sinh\left(n\pi\frac{b}{a}\right)}\left[V_2 \sinh\left(n\pi\frac{x}{a}\right)-V_1 \sinh\left(n\pi\left(\frac{x}{a}-\frac{b}{a}\right)\right)\right]\sin\left(n\pi\frac{y}{a}\right)$$

I’ve made a nice model of this potential for some rather specific values of the above parameters.

Suppose we have a box held at potentials V3 and V4 but pointed along the x axis. Then:

$$V(x,y)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\left[V_3\frac{\sinh\left(\frac{n\pi}{a}(y-b)\right)}{\sinh\left(-\frac{n\pi b}{a}\right)}+V_4\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\right]\sin\left(\frac{n\pi}{a}x\right)$$

Suppose we have a box held at potentials V1V2V3V4. The solution is then:

$$V(x,y)=\sum_{n=1,3,5,\ldots}^{\infty}\frac{4}{n\pi}\left[\left[V_3\frac{\sinh\left(\frac{n\pi}{a}(y-b)\right)}{\sinh\left(-\frac{n\pi b}{a}\right)}+V_4\frac{\sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\right]\sin\left(\frac{n\pi}{a}x\right)+\left[V_1\frac{\sinh\left(\frac{n\pi}{a}(x-b)\right)}{\sinh\left(-\frac{n\pi b}{a}\right)}+V_2\frac{\sinh\left(\frac{n\pi x}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}\right]\sin\left(\frac{n\pi}{a}y\right)\right]$$

Nasty, but it’s really the general solution for a box with 4 potentials!

A 3-d laplace problem:
This is example 3.5 in your text.
we need to solve:

$$\vec{\nabla}^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

we separate as before, assuming a solution of the form

$$V = XYZ$$

We then can obtain:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = 0$$

We have three separation constants here:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1$$

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2$$

$$\frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = C_3$$

with $C_1 + C_2 + C_3 = 0$

Here are the bc:
V=0@y=0, V=0@y=a, V=0@z=0,V=0@z=b,V=0@x=infinity,V=Vo(x,y)@x=0
we therefore want oscillatory solutions in the y and z direction. We'll thus choose these separation constants to be negative. Thus:

$$C_2 = -k^2; C_3 = -p^2; C_1 = k^2 + p^2$$

In the x-direction, we have exponentially damped , increasing solutions. We thus have:

$$V(x,y,z) = \left(Ae^{+\sqrt{k^2+p^2}\,x} + Be^{-\sqrt{k^2+p^2}\,x}\right)\left(C\sin(ky) + D\cos(ky)\right)\left(E\sin(pz) + F\cos(pz)\right)$$

I can pick out constants that need to vanish pretty easily. They are:
A, D and F. Combining the remaining constants gives:

$$V(x,y,z) = Ce^{-\sqrt{k^2+p^2}\,x}\sin(ky)\sin(pz)$$

We use two of the other boundary conditions to evaluate k and p:

$$ka = n\pi \Rightarrow k = \frac{n\pi}{a}$$

$$pb = m\pi \Rightarrow p = \frac{m\pi}{b}$$

The general solution is a double sum:

$$V_{nm}(x,y,z) = C_{nm}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

We thus have:

$$V(x,y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{nm}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\,x}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

Let me show you how to fit up the remaining boundary condition. At x=0, we have:

$$V_0(y,z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{nm}\sin\left(\frac{n\pi}{a}y\right)\sin\left(\frac{m\pi}{b}z\right)$$

You will evaluate this with an expanded version of Fourier’s trick:

$$\int_{y=0}^{y=a}\int_{z=0}^{z=b} V_0\left(y,z\right)\sin\left(\tfrac{n'\pi}{a}y\right)\sin\left(\tfrac{m'\pi}{b}z\right)dydz=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}\int_{y=0}^{y=a}\int_{z=0}^{z=b}\sin\left(\tfrac{n\pi}{a}y\right)\sin\left(\tfrac{m\pi}{b}z\right)\sin\left(\tfrac{n'\pi}{a}y\right)\sin\left(\tfrac{m'\pi}{b}z\right)dydz$$

We know how to evaluate this on the right hand side. This gives:

$$\int_{y=0}^{y=a}\int_{z=0}^{z=b} V_0\left(y,z\right)\sin\left(\tfrac{n'\pi}{a}y\right)\sin\left(\tfrac{m'\pi}{b}z\right)dydz=C_{n'm'}\tfrac{a}{2}\tfrac{b}{2}$$

Let’s eliminate the primes to write:

$$C_{nm}=\tfrac{4}{ab}\int_{y=0}^{y=a}\int_{z=0}^{z=b} V_0\left(y,z\right)\sin\left(\tfrac{n\pi}{a}y\right)\sin\left(\tfrac{m\pi}{b}z\right)dydz$$

This is still quite general at this point. Let’s work now the specific problem of the potential being constant on that face. We then have:

$$C_{nm}=\tfrac{4V_0}{ab}\int_{y=0}^{y=a}\int_{z=0}^{z=b}\sin\left(\tfrac{n\pi}{a}y\right)\sin\left(\tfrac{m\pi}{b}z\right)dydz$$

We know how to evaluate each of these integrals also. The result is:

$$C_{nm}=\begin{bmatrix} 0 \text{ if n or m is even} \\ \tfrac{16V_0}{\pi nm}\text{ is n and m are odd}\end{bmatrix}$$

It is now possible to write the general solution:

$$V_0\left(y,z\right)=\frac{16V_0}{\pi}\sum_{n=1,3,5,...}^{\infty}\sum_{m=1,3,5,...}^{\infty}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}\;x}\;\frac{\sin\left(\frac{n\pi}{a}y\right)}{n}\frac{\sin\left(\frac{m\pi}{b}z\right)}{m}$$

Your author suggests that a reasonable approximation would involve keeping only the first few terms, however for more precise numerical calculations, you will want to formulate some more extended method, I believe.

Let’s go on now to solutions in symmetries other than rectangular.

Spherical Coordinates3.3.2

We are going to follow the lead of your author here and work only with problems that have azimuthal symmetry. If you need to break this symmetry, I’ll refer you to “Classical Electromagnetic Radiation”, Second Edition by Marion and Heald page 65 where the problem is given a more general treatment. The treatment that we’ll give this problem will work in many situations and the more general solution is really not all that much more complicated. One note about the reference … another set of electrostatic units is defined and used in that text.

If you refer to the useful page, you will find the Laplacian in spherical coordinates given as:

$$\vec{\nabla}^2T=\tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial T}{\partial r}\right)+\tfrac{1}{r^2\sin(\theta)}\tfrac{\partial}{\partial\theta}\left(\sin\left(\theta\right)\tfrac{\partial T}{\partial\theta}\right)+\tfrac{1}{r^2\sin^2(\theta)}\tfrac{\partial^2T}{\partial\varphi^2}$$

In the present treatment, this reduces to:

$$\vec{\nabla}^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial V}{\partial \theta}\right)$$

Laplace's equation is:

$$\vec{\nabla}^2 V = 0$$

We thus need to solve:

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial V}{\partial \theta}\right) = 0$$

We will separate this as before:

$$V = R\Theta$$

We thus have the simplified result:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2 \frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{\partial}{\partial \theta}\left(\sin(\theta)\frac{\partial \Theta}{\partial \theta}\right) = 0$$

Because people have worked with this before, we know how to most effectively choose the separation constant which is of the form l(l+1). We'll choose it so that:

$$\frac{1}{R}\frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) = l(l+1)$$

$$\frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

Sorry that the fonts for 1 and l are so close.

We can write the solutions to the R equation:

$$R(r) = Ar^{l} + \frac{B}{r^{l+1}}$$

The angular equation is more complicated.

$$\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = -l(l+1)\sin(\theta)\Theta$$

These solutions are, however, well know to physicists … the solutions most useful are known as Legendre Polynomials:

$$\Theta(\theta) = P_l(\cos(\theta))$$

And the polynomials are most easily define by the Rodrigues formula:

$$P_l(x) = \frac{1}{2!l!}\left(\frac{d}{dx}\right)^{l}\left(x^2 - 1\right)^{l}$$

There is, of course, an entire other set of solutions but normally these are not used except for special geometries where, for example, the z-axis can not be reached, because of their divergent properties along the z-axis.

You can find out a lot of information about the Legendre polynomials at:
http://mathworld.wolfram.com/LegendrePolynomial.html

Here are the first few polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$$

$$P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$$

$$P_4(x) = \frac{1}{8}\left(35x^4 - 30x^2 + 3\right)$$

$$P_5(x) = \frac{1}{8}\left(63x^5 - 70x^3 + 15x\right)$$

There are some important properties that you need to know about the Legendre Polynomials:

$$\int_{-1}^{+1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$

You can evaluate other integrals involving Legendre Polynomials by reference to the URL given previously.

We thus can write the most general partial solution to V in this case:

$$V_l(r,\theta) = \left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

The solution satisfying a particular set of boundary conditions will then be:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

I am going to work through the examples in your text.

Example 3.6

The potential is Vo(θ) on a sphere of radius R. Find the potential inside the sphere.

The general solution is:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^l + \frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

We can immediately say that the B's are all zero since we don't want a divergence at the center of the sphere. Thus, the solution reduces to:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^l P_l(\cos(\theta))$$

On the surface of the sphere, we require:

$$V(R,\theta) = V_0(\theta)$$

We thus have:

$$V_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l(\cos(\theta))$$

We now employ Fourier's trick:

$$\int_{\theta=0}^{\theta=\pi} V_0(\theta)P_{l'}(\cos(\theta))\left[\sin(\theta)d\theta\right] = \sum_{l=0}^{\infty} A_l R^l \int_{\theta=0}^{\theta=\pi} P_l(\cos(\theta))P_{l'}(\cos(\theta))\left[\sin(\theta)d\theta\right]$$

Look at the orthonormality condition above:

$$\int_{-1}^{+1} P_n(x)P_m(x)\,dx = \frac{2}{2n+1}\delta_{mn}$$

This can also be written as:

$$\int_{\theta=0}^{\theta=\pi} P_l(\cos(\theta))P_{l'}(\cos(\theta))\left[\sin(\theta)d\theta\right] = \frac{2}{2l'+1}\delta_{ll'}$$

(you'll notice a shift in the limits for this definition. It's ok here)

We thus see that the integral reduces to become:

$$\int_{\theta=0}^{\theta=\pi} V_0(\theta) P_{l'}(\cos(\theta))\left[\sin(\theta)d\theta\right] = \sum_{l=0}^{\infty} A_l R^{l} \frac{2}{2l'+1}\delta_{ll'} = \frac{2}{2l'+1} A_{l'} R^{l'}$$

We can now write the general solution for the A’s:

$$A_l = \frac{2l+1}{2R^{l}} \int_{\theta=0}^{\theta=\pi} V_0(\theta) P_l(\cos(\theta))\left[\sin(\theta)d\theta\right]$$

Your author refers to the “eyeball” method of solution for the A’s. let’s look at what he means.

Suppose that on the sphere, the potential is given by:

$$V_0(\theta) = k\sin^2\left(\frac{\theta}{2}\right)$$

Since $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}\left(1-\cos(\theta)\right)$

we have:

$$V_0(\theta) = \frac{k}{2}\left(1-\cos(\theta)\right) = \frac{k}{2}\left(P_0(\cos(\theta)) - P_1(\cos(\theta))\right)$$

This means that we’re only going to get two possibilities for the A’s:

(1) L=0:

$$A_0 = \frac{k}{4R^0} \int_{\theta=0}^{\theta=\pi} P_0(\cos(\theta)) P_0(\cos(\theta))\left[\sin(\theta)d\theta\right] = \frac{k}{2R^0} = \frac{k}{2}$$

(2) L=1:

$$A_1 = -\frac{k}{2}\frac{2+1}{2R} \int_{\theta=0}^{\theta=\pi} P_1(\cos(\theta)) P_1(\cos(\theta))\left[\sin(\theta)d\theta\right] = -\frac{k}{2}\frac{2+1}{2R}\frac{2}{2+1} = -\frac{k}{2R}$$

The solution to this problem is then:

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l r^{l} P_l(\cos(\theta)) = \frac{k}{2} - k\frac{r}{2R}\cos(\theta) = \frac{k}{2}\left(1-\frac{r}{R}\cos(\theta)\right)$$

We can write this more simply: since $z = r\cos(\theta)$

we have:

$$V(r,\theta) = V(z) = \frac{k}{2}\left(1-\frac{z}{R}\right)$$

which clearly shows a linear decrease in V as you go from z=-R to z=+R. I’m not sure why your author did not do this final step.

Let’s now go ahead and solve for the region outside the sphere. In this case, we need a different set of solutions. We had in general:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^{l} + \frac{B_l}{r^{l+1}}\right) P_l(\cos(\theta))$$

Here, however, the A’s must go to zero and not the B’s. Thus:

$$V(r,\theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos(\theta))$$

Notice the typo in eq. 3.72 and in the equation following it.

As before, the angular dependence is going to be pretty straight-forward.

$$\frac{B_l}{R^{l+1}} = \begin{bmatrix} 0 \text{ if } l \neq 0 \text{ or } l \neq 1 \\ \frac{k}{2} \text{if } l = 0 \\ -\frac{k}{2} \text{if } l = 1 \end{bmatrix} \Rightarrow B_l = \begin{bmatrix} 0 \text{ if } l \neq 0 \text{ or } l \neq 1 \\ R\frac{k}{2} \text{if } l = 0 \\ -R^2\frac{k}{2} \text{if } l = 1 \end{bmatrix}$$

The general solution is then:

$$V(r,\theta) = \frac{k}{2}\frac{R}{r} - \frac{k}{2}\frac{R^2}{r^2}P_1(\cos(\theta)) = \frac{k}{2}\frac{R}{r}\left(1 - \frac{R}{r}P_1(\cos(\theta))\right)$$

Let's now look at another application of this formalism.

Example 3.8

An uncharged conducting sphere of radius R is placed in an otherwise uniform electric field directed along the z axis. Find the potential in the region outside the sphere. You may assume the sphere is at zero potential.

In this case, the potential at infinity will diverge as -Ez+C. This kind-of means we're going to have to be careful what we throw away in the constants. We have as boundary conditions:

$$V = -Ez \text{ as } r \to \infty$$

$$V = 0 \text{ for r=R}$$

The general solution from Laplace's equation is:

$$V(r,\theta) = \sum_{l=0}^{\infty}\left(A_l r^1 + \frac{B_1}{r^{l+1}}\right)P_l(\cos(\theta))$$

At r=R, we require V=0. Thus, we have:

$$\left(A_l R^1 + \frac{B_1}{R^{l+1}}\right) = 0 \Rightarrow B_l = -A_l R^{21+1}$$

When r=R and θ=0, the potential must be equal to zero. This means that the L=0 term will not be present.

The boundary condition at infinity means that only the L=1 term will be present in the final solution. We thus have:

$$V(r,\theta) = A_1\left(r^1 - \frac{R^3}{r^2}\right)\cos(\theta)$$

The final step is to fit A1: at infinity, we find:

A1=-E. Thus,

$$V(r,\theta) = -E_0\left(r^1 - \frac{R^3}{r^2}\right)\cos(\theta)$$

$$V\left(r,\theta\right)=-E_0\left(r^1-\frac{R^3}{r^2}\right)\cos\left(\theta\right)$$

You can easily find now the electric field:

$$\vec{E}=-\vec{\nabla}V=-\vec{\nabla}T=-\frac{\partial V}{\partial r}\hat{r}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}$$

Thus:

$$\frac{\partial V}{\partial r}=-E_0\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\left(\theta\right)$$

$$\frac{\partial V}{\partial\theta}=E_0\left(r-\frac{R^3}{r^2}\right)\sin\left(\theta\right)$$

we then have the electric field given by:

$$\vec{E}=E_0\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\left(\theta\right)\hat{r}-E_0\left(1-\left(\frac{R}{r}\right)^3\right)\sin\left(\theta\right)\hat{\theta}$$

We can also find the induced surface charge density:

$$\sigma\left(\theta\right)=-\varepsilon_0\left.\frac{\partial V}{\partial r}\right|_{r=R}=\varepsilon_0E_0\left.\left(1+2\left(\frac{R}{r}\right)^3\right)\cos\left(\theta\right)\right|_{r=R}=3\varepsilon_0E_0\cos\left(\theta\right)$$

Now it would not be correct to place two capacitor plate near the sphere (using this solution) and then try to determine a new potential on the capacitor plates. This is because we would be violating the assumed boundary condition that we used above.

We can, however, calculate the energy density of the field. This is given by:

$$u=\frac{1}{2}\varepsilon_0\vec{E}\bullet\vec{E}$$

Let's do this:

$$u=\frac{1}{2}\varepsilon_0E_0^2\left[\cos^2\left(\theta\right)\left(1+4\left(\frac{R}{r}\right)^3+4\left(\frac{R}{r}\right)^6\right)+\sin^2\left(\theta\right)\left(1-4\left(\frac{R}{r}\right)^3+4\left(\frac{R}{r}\right)^6\right)\right]$$

$$u=\frac{1}{2}\varepsilon_0E_0^2\left[1+4\left(\frac{R}{r}\right)^6+4\left(\frac{R}{r}\right)^3\left[\cos^2\left(\theta\right)-\sin^2\left(\theta\right)\right]\right]$$

$$u=\frac{1}{2}\varepsilon_0E_0^2\left[1+4\left(\frac{R}{r}\right)^6-4\left(\frac{R}{r}\right)^3+8\left(\frac{R}{r}\right)^3\cos^2\left(\theta\right)\right]$$

$$u=\frac{1}{2}\varepsilon_0E_0^2\left[1+4\left(\frac{R}{r}\right)^6-4\left(\frac{R}{r}\right)^3+8\left(\frac{R}{r}\right)^3\left(\frac{z}{r}\right)^2\right]$$

This gives a function which is fairly smooth and drops off rapidly as you move away from the sphere.