A 3-d laplace problem: 2019

This is example 3.5 in your text.

we need to solve:

$$\vec{\nabla}^2 V = 0 \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

we separate as before, assuming a solution of the form

$$V = X(x)Y(y)Z(z)$$

We then can obtain:

$$\frac{1}{x}\frac{\partial^2 X}{\partial x^2} + \frac{1}{y}\frac{\partial^2 Y}{\partial y^2} + \frac{1}{z}\frac{\partial^2 Z}{\partial z^2} = 0$$

We have three separation constants here:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1 : \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2 : \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = C_3 \quad \text{with} \quad C_1 + C_2 + C_3 = 0$$

Here are the boundary conditions:

$$V = 0 @ y = 0 ; V = 0 @ y = a$$

$$V = 0 @ z = 0 ; V = 0 @ z = b$$

$$V = 0 @ x \rightarrow \infty ; V = V_0(y, z) @ x = 0$$

we therefore want oscillatory solutions in the y and z direction. We'll thus choose these separation constants to be negative. Thus:

$$C_2 = -k^2 ; C_3 = -p^2 ; C_1 = k^2 + p^2$$

In the x-direction, we have exponentially damped , increasing solutions. We thus have:

$$V(x, y, z) = \left(A e^{+\sqrt{k^2+p^2}x} + B e^{-\sqrt{k^2+p^2}x}\right)\left(C \sin(ky) + D \cos(ky)\right)\left(E \sin(pz) + F \cos(pz)\right)$$

I can pick out constants that need to vanish pretty easily. They are: A, D and F. Combining the remaining constants gives:

$$V(x, y, z) = C e^{-\sqrt{k^2+p^2}x} \sin(ky) \sin(pz)$$

We use two of the other boundary conditions to evaluate k and p:

$$ka = n\pi \Rightarrow k = \frac{n\pi}{a} : pb = m\pi \Rightarrow p = \frac{m\pi}{b}$$

The general solution is a double sum:

$$V_{nm}(x, y, z) = C_{nm} e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}x} \sin\left(\frac{n\pi y}{a}\right) \sin\left(\frac{m\pi z}{b}\right)$$

We thus have:

$$V_{nm}(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}x}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

Let me show you how to fit up the remaining boundary condition. At x=0, we have:

$$V_{nm}(x,y,z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{nm}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

You will evaluate this with an expanded version of Fourier's trick:

$$\int_{y=0}^{y=a}\int_{z=0}^{z=b}dy\,dz\,V_{nm}(0,y,z)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)=$$

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int_{y=0}^{y=a}\int_{z=0}^{z=b}dy\,dz\,C_{nm}\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)$$

We know how to evaluate this on the right hand side. This gives:

$$\int_{y=0}^{y=a}\int_{z=0}^{z=b}dy\,dz\,V_{nm}(0,y,z)\sin\left(\frac{n'\pi y}{a}\right)\sin\left(\frac{m'\pi z}{b}\right)=C_{n'm'}\frac{a}{2}\frac{b}{2}$$

Let's eliminate the primes to write:

$$C_{nm}=\frac{4}{ab}\int_{y=0}^{y=a}\int_{z=0}^{z=b}dy\,dz\,V_{nm}(0,y,z)\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

This is still quite general at this point. Let's work now the specific problem of the potential being constant on that face. We then have:

$$C_{nm}=\frac{4V_0}{ab}\int_{y=0}^{y=a}\int_{z=0}^{z=b}dy\,dz\sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi z}{b}\right)$$

We know how to evaluate each of these integrals also. The result is:

$$C_{nm}=\begin{bmatrix} 0 & \text{if n or m is even} \\ \frac{16V_0}{\pi nm} & \text{if n and m are odd}\end{bmatrix}$$

It is now possible to write the general solution:

$$V(x,y,z)=\frac{16V_0}{\pi}\sum_{n=1,3,5,...}^{\infty}\sum_{m=1,3,5,...}^{\infty}e^{-\pi\sqrt{\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2}}\frac{\sin\left(\frac{n\pi y}{a}\right)}{n}\frac{\sin\left(\frac{m\pi z}{b}\right)}{m}$$

Your author suggests that a reasonable approximation would involve keeping only the first few terms, however for more precise numerical calculations, you will want to formulate some more extended method, I believe.

Let's go on now to solutions in symmetries other than rectangular.