The 3 D delta function

In 3-dimensions, the delta function would appear as:

$$\delta^3(\vec{r})=\delta(x)\delta(y)\delta(z) \quad .$$

where $\vec{r}\equiv x\,\hat{x}+y\,\hat{y}+z\,\hat{z}$ is the position vector. The 3-d delta function is defined by:

$$\delta^3(\vec{r})=\begin{cases} 0 & \text{if } x,y,z \neq (0,0,0) \\ \infty & \text{if } (x,y,z)=(0,0,0) \end{cases} \quad \text{with}$$

$$\int_{\text{all space}} \delta^3(\vec{r})\,d^3r=\int_{x=-\infty}^{x=+\infty}\int_{y=-\infty}^{y=+\infty}\int_{z=-\infty}^{z=+\infty}\delta(x)\delta(y)\delta(z)\,dx\,dy\,dz=1 \quad .$$

We also then have the shifting property:

$$\int_{\text{all space}} f(\vec{r})\,\delta^3(\vec{r}-\vec{a})\,d^3r=f(\vec{a}) \quad .$$

In terms of the earlier divergence, we can now identify the 3-d delta function provided it satisfies what we found earlier:

$$4\pi\delta^3(\vec{r})=\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) \quad .$$

more generally, in terms of r_{ip} we have:

$$4\pi\delta^3(\vec{r}_{ip})=\vec{\nabla}_p\cdot\left(\frac{\hat{r}_{ip}}{r_{ip}^2}\right)$$

where the divergence operator $\vec{\nabla}_p$ is written so as to emphasize differentiation with respect to the p coordinates and $\vec{r}_{ip}\equiv\vec{r}_p-\vec{r}_i$.

It is probably important to emphasize here that you should not forget the “strength” factor of 4π when you are working in 3-D space with the divergence which is easy to forget.

Something like this might work: “if you have 3, you need 4 when diverging.”

To emphasize: the delta function is not: $\delta^3(\vec{r}_{ip})\neq\vec{\nabla}_p\cdot\left(\frac{\hat{r}_{ip}}{r_{ip}^2}\right)$!!!

I am going to also follow the examples from your author now.

Example 1.16: Evaluate

$$J \equiv \oiiint (r^2+2)\,\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) d^3 r$$

where the volume of interest is a sphere of radius R centered at the origin.

Solution 1: I'll only say look at your author's solution 2.

According to the definition of the 3-D Dirac function, we have:

$$J = \oiiint (r^2+2)\,4\pi\delta^3(\vec{r})\,d^3r = 4\pi(0+2) = 8\pi \quad .$$

Problem 1.46

(a) Write an expression for the electric charge density $\rho(\vec{r})$ of a point charge . Make sure that the volume integral of ρ equals q.

This comes actually right from the definition of the 3-D delta function:

$$q\delta^3(\vec{r}-\vec{r}_i) = \rho(\vec{r}): \int_{\text{all charges}} Q\delta^3(\vec{r}-\vec{r}_i)\,d^3r_i = \int_{\text{all space}} Q\delta^3(\vec{r}-\vec{r}_i)\,d^3r = Q(\vec{r}_i) = Q$$

Remember: An important test of a charge density is does the integral over all charges give the total charge Q?

(b) What is the charge density of an electric dipole, consisting of a point charge -q at the origin and a point charge +q at $\vec{a}$?

$$\rho\vec{r}_i = -q\delta^3(\vec{r}-\vec{0}) + q\delta^3(\vec{r}-\vec{a})$$

(c) What is the charge density of a uniform, infinitesimally thin shell of radius R and total charge Q, centered at the origin? [Beware: the integral over all space must equal Q.]

Here, the basis of the answer is:

$$\rho(\vec{r}) = \text{constant } x\delta^3(\vec{r}-\vec{R})$$

to evaluate the constant, we need to do the integral:

$$\oiiint_{\text{all space}} \rho(\vec{r})\,d^3r = \int_{r=0}^{r=+\infty}\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi} C\delta^3(\vec{r}-\vec{R})\,r^2\,dr\sin\theta\,d\theta\,d\phi = 4\pi C R^2 = Q$$

$$\Rightarrow C = \frac{Q}{4\pi R^2} \Rightarrow \rho(\vec{r}) = \frac{Q}{4\pi R^2}\delta^3(\vec{r}-\vec{R})$$

Note: if you had N charges q_i , located at $\vec{r}_i; i=1,2,3,\ldots,N$, the charge density is

$$\rho(\vec{r}) = \sum_{i=1}^{i=N} q_i\delta^3(\vec{r}-\vec{r}_i)$$

Also worth noting that the dimensions of the 3-D delta function are inverse volume. This is all important because I have now told you how in general to describe lots of charge distributions in a general way; something that is really useful.

Problem 1.48 Evaluate the integral

$$J \equiv \oiiint e^{-r}\left(\nabla \cdot \frac{\hat{r}}{r^2}\right) d^3 r$$

where the volume is a sphere of radius R, centered at the origin (I'll do with the 3-D delta function, you do with parts). Reference Example 1.16.

$$\nabla \cdot \left(\frac{\hat{r}}{r^2}\right) = 4\pi \delta^3(\vec{r})$$

so: $J \equiv \oiiint e^{-r} 4\pi \delta^3(\vec{r} - \vec{0}) d^3 r = \int_{r=0}^{r=+\infty} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} e^{-r} 4\pi \delta^3(\vec{r} - \vec{0}) r^2 dr \sin\theta \, d\theta \, d\phi = 0$