Electric Fields in Matter

4.1 polarization

When matter is placed in an external electric field, the electric field is usually capable of displacing the positive and negative charge centers in different directions. This leads to the the concept of atomic polarization and normally the polarization is in the same direction as E. This is expressed by

$$\vec{p}=\alpha\vec{E}$$

α is the atomic polarizability of the species and the dimensions are determined by:

$$p:[qL]\,E:[F/q]:[\alpha]=\frac{[qL]}{[F/q]}=\frac{[q^2L]}{[ML/t^2]}=\frac{[q^2t^2]}{[M]}$$

We could write this in other terms:

$$[qL]=[\alpha]k\frac{q}{L^2}\Rightarrow[\alpha]=\frac{[L^3]}{k}\Rightarrow\frac{[\alpha]}{[4\pi\epsilon_0]}=L^3$$

The polarizability tensor

In general, in molecules, the induced polarization does not need to be in the same direction as E; or is not the same in all directions. You might get away with, in many cases, writing the polarization with parallel and perpendicular components :

$$\vec{p}=\alpha_{\parallel}\vec{E}_{\parallel}+\alpha_{\perp}\vec{E}_{\perp}$$

Where the components are relative to some particular molecular axis. In general, however, the polarization would be given by a tensor:

$$\vec{p}=\begin{bmatrix}\alpha_{xx} & \alpha_{xy} & \alpha_{xz}\\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz}\\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz}\end{bmatrix}\cdot\begin{bmatrix}E_x\hat{x}\\ E_y\hat{y}\\ E_z\hat{z}\end{bmatrix}$$

According to your author, it is always possible to choose principle axes so that off diagonal terms vanish (E would need to be properly translated to these coordinates). This would involve diagonalizing the matrix.

For now, we will assume that the displacement field is given by:

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}\quad\text{and}\quad\oint\!\!\oint\vec{D}\cdot d\vec{A}=Q_{free_{enclosed}}\quad\text{and}\quad\vec{\nabla}\cdot\vec{D}=\rho_{free}$$

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The polarization is defined as the net dipole moment per unit volume.

Field of a polarized object : for a single dipole, we had $V(r_p)=\frac{1}{4\pi\epsilon_0}\frac{\vec{p}\cdot\hat{r}_{ip}}{r_{ip}^2}$.

In terms of the polarization, we have:

$$V(r_p)=\frac{1}{4\pi\epsilon_0}\int_{\text{all } q_i}\left[\frac{\vec{P}\cdot\hat{r}_{ip}}{r_{ip}^2}d^3r_i\right]$$

But remember that this expression is approximate. It works most times.

$$\vec{\nabla}_i\left(\frac{1}{r_{ip}}\right)=\frac{\hat{r}_{ip}}{r_{ip}^2}\Rightarrow V(r_p)=\frac{1}{4\pi\epsilon_0}\int_{\text{all } q_i}\vec{P}\cdot\vec{\nabla}_i\left(\frac{1}{r_{ip}}\right)d^3r_i$$

$$\vec{\nabla}_i\cdot\left(\frac{\vec{P}}{r_{ip}}\right)=\frac{1}{r_{ip}}\vec{\nabla}_i\cdot\vec{P}+\vec{P}\cdot\vec{\nabla}\cdot\left(\frac{1}{r_{ip}}\right)\Rightarrow V(r_{ip})=\frac{1}{4\pi\epsilon_0}\int_{\text{all } q_i}\frac{1}{r_{ip}}\vec{\nabla}\cdot\vec{P}d^3r_i-\frac{1}{4\pi\epsilon_0}\int_{\text{all } q_i}\vec{\nabla}\cdot\left(\frac{\vec{P}}{r_{ip}}\right)d^3r$$

Use the Divergence Theorem, $\oiiint(\vec{\nabla}\cdot\vec{T})d^3r=\oiint\vec{T}\cdot d\vec{A}$ on the second term to get:

$$V(r_{ip})=\frac{1}{4\pi\epsilon_0}\int_{\text{surface}}\left(\frac{\vec{P}\cdot\hat{n}}{r_{ip}}\right)d^2r-\frac{1}{4\pi\epsilon_0}\int_{\text{all } q_i}\frac{1}{r_{ip}}\vec{\nabla}\cdot\vec{P}d^3r_i$$

The first term is a surface charge density: $\sigma_b\equiv\vec{P}\cdot\hat{n}$.

The second term is a volume charge density: $\rho_b\equiv-\vec{\nabla}\cdot\vec{P}$

Which means that the potential (and thus the field) produced by a polarized object is the same as would be produced by a surface charge density superimposed with a volume charge density given the the definitions above. It is important to keep in mind that a very special case of this is when **P is uniform**. In that case, the volume charge density is zero.

Example 4.2

Find the electric field on a sphere of radius a which has a uniform polarization given by $\vec{P}=P\hat{z}$.

Since P is uniform, there is no corresponding volume charge density. The surface charge density is given by:

$$\vec{P}\cdot\hat{n}=\vec{P}\cdot\hat{r}_i=P\cos(\theta)$$

Let's just step back a few minutes to make sure we understand how to get the solution.

A sphere of radius a has a surface charge density $\sigma(\theta)$ glued on its surface. Find the potential everywhere.

Inside: $V(r,\theta)=\sum_{n=0}^{\infty}A_n r^n P_n(\cos(\theta))$

Outside: $V(r,\theta)=\sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos(\theta))$

At the surface, the potentials are continuous.

$$\sum_{n=0}^{\infty}A_n a^n P_n(\cos(\theta))=\sum_{n=0}^{\infty}\frac{B_n}{a^{n+1}}P_n(\cos(\theta))$$

This gives the connection by equating the orders of the Legendre polynomials:

$$A_n a^n = \frac{B_n}{a^{n+1}} \Rightarrow B_n = A_n a^{2n+1}$$

The discontinuity in the derivative of the potential at the boundary is given by:

$$\left(\frac{\partial V_{out}}{\partial r} - \frac{\partial V_{in}}{\partial r}\right)_{r=a} = \frac{-\sigma(\theta)}{\epsilon_0}$$

$$-\sum_{n=0}^{\infty}(n+1)\frac{B_n}{a^{n+2}}P_n(\cos(\theta)) - \sum_{n=0}^{\infty} n A_n a^{n-1} P_n(\cos(\theta)) = \frac{-\sigma_0(\theta)}{\epsilon_0}$$

So: $\displaystyle\sum_{n=0}^{\infty}(2n+1)A_n a^{n-1}P_n(\cos(\theta)) = \frac{\sigma_0(\theta)}{\epsilon_0}$

The coefficients are then determined by:

$$\int_0^{\pi} P_m(\cos(\theta))\sin(\theta)\,d\theta \sum_{n=0}^{\infty}(2n+1)A_n a^{n-1}P_n(\cos(\theta)) = \int_0^{\pi}\frac{\sigma_0(\theta)}{\epsilon_0}P_m(\cos(\theta))\sin(\theta)\,d\theta$$

$$\Rightarrow (2m+1)A_m a^{m-1}\frac{2}{2m+1} = \frac{1}{\epsilon_0}\int_0^{\pi}\sigma_0(\theta)P_m(\cos(\theta))\sin(\theta)\,d\theta$$

$$\Rightarrow A_m = \frac{1}{2\epsilon_0 a^{m-1}}\int_0^{\pi}\sigma_0(\theta)P_m(\cos(\theta))\sin(\theta)\,d\theta$$

As an example, if $\sigma_0(\theta) = c\cos(\theta) = cP_1(\cos(\theta))$

and $\displaystyle A_1 = \frac{1}{2\epsilon_0}c\int_0^{\pi}P_1(\theta)P_1(\theta)\sin(\theta)\,d(\theta)$

But: since $\displaystyle\int_{\theta=0}^{\theta=\pi}P_l(\cos(\theta))P_{l'}(\cos(\theta))\sin(\theta)\,d\theta = \frac{2}{2l'+1}\delta_{ll'}$

So we then have

$$A_1 = \frac{1}{2\epsilon_0}c\frac{2}{3} = \frac{c}{3\epsilon_0}$$

If this charge is due to polarization, then we can say that c=P.

So inside the sphere: $\displaystyle V(r,\theta) = \sum_{n=0}^{\infty}A_n r^n P_n(\cos(\theta)) \Rightarrow V = \frac{c}{3\epsilon_0}r\cos(\theta); r<a$

Outside the sphere: $\displaystyle V(r,\theta) = \sum_{n=0}^{\infty}A_n\frac{a^{2n+1}}{r^{n+1}}P_n(\cos(\theta)) \Rightarrow V = \frac{c}{3\epsilon_0}\frac{a^3}{r^2}\cos(\theta); r>a$

Compare to the problem of a metal sphere in a uniform electric field. You see that the field inside the sphere cancels exactly the external field. Outside the sphere, you can find the field by :

$$\vec{E}=-\vec{\nabla}V \quad .$$

$$V_{inside}=\frac{c}{3\epsilon_0}r\cos(\theta);r<a$$

$$V_{outside}=\frac{c}{3\epsilon_0}\frac{a^3}{r^2}\cos(\theta);r>a$$

Let's go ahead and calculate E

In spherical coordinates: $\vec{\nabla}T=\frac{\partial T}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial T}{\partial\theta}\hat{\theta}+\frac{1}{r\sin(\theta)}\frac{\partial T}{\partial\phi}\hat{\phi}$

inside: $\frac{\partial V}{\partial r}=\frac{c}{3\epsilon_0}\cos(\theta):\frac{1}{r}\frac{\partial V}{\partial\theta}=\frac{c}{3\epsilon_0}(-\sin(\theta))\Rightarrow\vec{E}_{inside}=\frac{c}{3\epsilon_0}[-\cos\theta\hat{r}+\sin(\theta)\hat{\theta}]$

outside: $\frac{\partial V}{\partial r}=\frac{ca^3}{3\epsilon_0}\left(\frac{-2\cos(\theta)}{r^3}\right):\frac{1}{r}\frac{\partial V}{\partial\theta}=\frac{ca^3}{3\epsilon_0}\frac{1}{r^3}(-\sin(\theta))\Rightarrow\vec{E}_{outside}=\frac{ca^3}{3\epsilon_0 r^3}[2\cos(\theta)\hat{r}+\sin(\theta)\hat{\theta}]$

Let's calculate D from $\vec{D}=\epsilon_0\vec{E}+\vec{P}$:

$$\vec{E}_{inside}=\frac{c}{3\epsilon_0}[-\cos\theta\hat{r}+\sin(\theta)\hat{\theta}]$$

Noting that $\hat{z}=\cos(\theta)\hat{r}-\sin(\theta)\hat{\theta}\Rightarrow\vec{E}_{inside}=\frac{-c}{3\epsilon_0}\hat{z}$

and $\vec{P}=c\hat{z}$ inside and $\vec{P}=0$ outside.

so

$$\vec{D}_{inside}=\frac{-P}{3}\hat{z}+P\hat{z}=\frac{2}{3}P\hat{z}:\vec{D}_{outside}=\frac{Pa^3}{3r^3}[2\cos(\theta)\hat{r}+\sin(\theta)\hat{\theta}]$$

Now let's calculate Q free:

In spherical coordinates, $\vec{\nabla}\cdot\vec{T}=\frac{1}{r^2}\frac{\partial(r^2T_r)}{\partial r}+\frac{1}{r\sin(\theta)}\frac{\partial(\sin(\theta)T_\theta)}{\partial\theta}+\frac{1}{r\sin(\theta)}\frac{\partial T_\phi}{\partial\phi}$.

$$\text{inside}:\vec{\nabla}\cdot\vec{D}=0$$

$$\text{outside}:\vec{\nabla}\cdot\vec{D}=\frac{Pa^3}{3r^2}\left[\frac{-2\cos(\theta)}{r^2}\right]+\frac{Pa^3}{3r^4\sin(\theta)}[2\sin(\theta)\cos(\theta)]=0$$

problem 4.10 r1 2016

A sphere of radius a carries a polarization $\vec{P}=c\vec{r}$. Find the surface and volume bound charge. Find the electric field inside and outside the sphere.

$$\rho_b=-\vec{\nabla}\cdot\vec{P}=-\vec{\nabla}\cdot c\,r\,\hat{r}$$

In spherical coordinates, $\vec{\nabla}\cdot\vec{T}=\frac{1}{r^2}\frac{\partial\left(r^2T_r\right)}{\partial r}+\frac{1}{r\sin(\theta)}\frac{\partial\left(\sin(\theta)T_\theta\right)}{\partial\theta}+\frac{1}{r\sin(\theta)}\frac{\partial T_\phi}{\partial\phi}$.

So, $\frac{1}{r^2}\frac{\partial\left(r^2cr\right)}{\partial r}=\frac{3cr^2}{r^2}\Rightarrow\rho_b=-3c$

$$\vec{P}\cdot\hat{n}=\vec{P}\cdot\hat{r}=cr\,\hat{r}\cdot\hat{r}\Rightarrow\sigma_b=cr$$

It is easy enough to find E from Gauss's law:

$$\Phi_E=\oiint\vec{E}\cdot d\vec{A}=\frac{Q_{enc}}{\epsilon_0}=\frac{1}{\epsilon_0}\oiiint\rho r^2dr\sin(\theta)d\theta d\phi$$

$$E(4\pi r^2)=\frac{-3c}{\epsilon_0}\left[\frac{4}{3}\pi r^3\right]\Rightarrow\vec{E}_{in}=\frac{-rc}{\epsilon_0}\hat{r}$$

$$\text{inside}:\Phi_E=\oiint\vec{E}\cdot d\vec{A}\Rightarrow\Phi_E=E\left(4\pi r^2\right)$$

$$Q_{enc}=\oiiint(-3c)r^2dr\sin(\theta)d\theta d\phi=-3C\left[\frac{4}{3}\pi r^3\right]\Rightarrow\vec{E}_{in}=-\frac{c}{\epsilon_0}\vec{r}=-\frac{\vec{P}}{\epsilon_0}$$

$$\text{outside}:\Phi_E=\oiint\vec{E}\cdot d\vec{A}\Rightarrow\Phi_E=E\left(4\pi r^2\right)$$

$$Q_{enc}=-3C\left[\frac{4}{3}\pi a^3\right]+c\,a^3(4\pi)=0\Rightarrow\vec{E}_{out}=0$$

Now let's calculate D using $\vec{D}=\epsilon_0\vec{E}+\vec{P}$:

$$\text{inside}:\vec{D}=-\vec{P}+\vec{P}=\vec{0}:\quad\text{outside}:\vec{D}=\vec{0}+\vec{0}=\vec{0}$$

Since there is no free charge, this is good.

Gauss's law in the presence of dielectrics

The total charge density is

$$\rho=\rho_b+\rho_f$$

Gauss's law says:

$$\epsilon_0\vec{\nabla}\cdot\vec{E}=\rho=\rho_b+\rho_f=-\vec{\nabla}\cdot\vec{P}+\rho_f\Rightarrow\vec{\nabla}\cdot\left(\epsilon_0\vec{E}+\vec{P}\right)=\rho_f$$

The “displacement” field is then defined by:

$$\vec{D}\equiv\epsilon_0\vec{E}+\vec{P}$$

And Gauss's law becomes:

$$\vec{\nabla}\cdot\vec{D}=\rho_f\leftrightarrow\oiint\vec{D}\cdot d\vec{A}=Q_{f,\ enclosed}$$

Ok, this is an approximate and a loose way to say it (and a bit incorrect):
D comes from free charges. E comes from charges.

Example 4.4

A wire surrounded by rubber to a radius a carries λ. Find D

Inside(s<a) And Outside(s>a): $\oiint\vec{D}\cdot d\vec{A}=Q_{f,\ enc}\Rightarrow D(2\pi s h)=\lambda h\Rightarrow\vec{D}=\frac{\lambda}{2\pi s}\hat{s}$

Outside (s>a): $\vec{P}=\vec{0}\Rightarrow\vec{D}=\epsilon_0\vec{E}\Rightarrow\vec{E}=\frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$ Inside (s<a): without P, E is unknown.

problem 4.15

$\vec{P}=\frac{c}{r}\hat{r}$: inner **Spherical** shell of radius a, outer shell of radius b.

bound charge: using: $\vec{\nabla}\cdot\vec{V}=\frac{1}{r^2}\frac{\partial(r^2V_r)}{\partial r}+\frac{1}{r\sin(\theta)}\frac{\partial(\sin(\theta)V_\theta)}{\partial\theta}+\frac{1}{r\sin(\theta)}\frac{\partial V_\phi}{\partial\phi}$

$$\rho_b=-\vec{\nabla}\cdot\vec{P}=-\left[\frac{1}{r^2}\frac{\partial(Cr)}{\partial r}\right]=\frac{-C}{r^2}:@r=a:\sigma_b=\vec{P}\cdot\hat{n}=\frac{C}{a}:@r=b:\sigma_b=\frac{-C}{b}$$

4.3.1 The deceptive parallel

There is no Coulomb's law for D ...

This does not exist: $\vec{D}\neq\frac{1}{4\pi}\int_{all\ q_{free}}\frac{\hat{r}}{r^2}\rho_{free}d^3r$.

Since

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}:\vec{\nabla}x\vec{D}=\vec{\nabla}x\vec{P}$$

and in general the curl of P does not vanish. However if the problem shows cylindrical, spherical or plane symmetry, the curl of P will vanish. Then you can use Gauss's law to calculate D via $\oiint\vec{D}\cdot d\vec{A}=Q_{free,\ enclosed}$