r21-1	Physics **250**: Worksheet 06	Name _______________

Review Worksheet

(1) 3 charges are specified as shown:
(#:q,x,y,z)=(1:1μC,-1,2,0),(2:-2μC,1,2,0),(3:4μC,0,0,0).
At a point in space p=(5,4,0):
(a) what are the vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_p, \hat{r}_{1p}, \hat{r}_{2p}, \hat{r}_{3p}$?
(b) What is the vector electric field at the point p?
(c) What is the electric potential at the point p?
(d) How much work did it take to assemble this charge distribution?

(2) In a certain region of space, the electric potential is $V = ax^2 + by^3$. What is the electric field in this region of space?

(3) Suppose you have a sphere of radius a which has a volume charge density $\rho(\vec{r}) = \rho_0 \frac{r}{a}$. What is the electric field inside and outside the sphere?

(4) A parallel plate capacitor has an area A=1000 m^2 and a separation of 1×10^{-3}m. How much work must be done to charge this capacitor up to a potential of 100V? How much charge is stored on the capacitor. What is the magnitude of the electric field inside the capacitor? If a material with a dielectric constant κ=200 is inserted between the plates of the capacitor (it completely fills the area), what is the capacitance?
If two of these capacitors (with the material) are connected in series, what is the equivalent capacitance. If two of these capacitors (with the material) are connected in parallel, what is the equivalent capacitance?

(5) Suppose you have two resistors, R_1 and R_2. What is the ratio of the series equivalent resistance to the parallel equivalent resistance if R_1=100Ω and R_2=300Ω?

(6) Calculate the capacitance of a solid conducting sphere of radius a.

(7) How much work is required to charge up an ideal coaxial capacitor of inner radius a and outer radius b and length h to a potential difference V?

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(1) 3 charges are specified as shown:
(#:q,x,y,z)=(1:1μC,-1,2,0),(2:-2μC,1,2,0),(3:4μC,0,0,0).
At a point in space p=(5,4,0):
(a) what are the vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \vec{r}_p, \hat{r}_{1p}, \hat{r}_{2p}, \hat{r}_{3p}$?

$$\vec{r}_1 = -1\hat{x}+2\hat{y}+0\hat{z} : \vec{r}_2 = 1\hat{x}+2\hat{y}+0\hat{z} : \vec{r}_3 = 0\hat{x}+0\hat{y}+0\hat{z} : \vec{r}_p = 5\hat{x}+4\hat{y}+0\hat{z}$$

$$\vec{r}_{1p} = \vec{r}_p - \vec{r}_1 = (5+1)\hat{x} + (4-2)\hat{y} + 0\hat{z} = 6\hat{x}+2\hat{y}$$
$$\vec{r}_{2p} = (5-1)\hat{x} + (4-2)\hat{y} = 4\hat{x}+2\hat{y}$$
$$\vec{r}_{3p} = (5-0)\hat{x} + (4-0)\hat{y} = 5\hat{x}+4\hat{y}$$

$$\hat{r}_{1p} = \frac{6}{\sqrt{36+4}}\hat{x} + \frac{2}{\sqrt{40}}\hat{y} = 0.949\hat{x} + 0.316\hat{y}$$
$$\hat{r}_{2p} = \frac{4}{\sqrt{16+4}}\hat{x} + \frac{2}{\sqrt{20}}\hat{y} = 0.894\hat{x} + 0.447\hat{y}$$
$$\hat{r}_{3p} = \frac{5}{\sqrt{25+16}}\hat{x} + \frac{4}{\sqrt{41}}\hat{y} = 0.781\hat{x} + 0.625\hat{y}$$

(b) What is the vector electric field at the point p?

Note that

$$\vec{E}_p = \sum_{i=1}^{i=n} k \frac{q_i}{|\vec{r}_p - \vec{r}_i|^2} \hat{r}_{ip} = \sum_{i=1}^{i=n} k \frac{q_i}{|\vec{r}_{ip}|^2} \hat{r}_{ip}$$

$$\vec{E} = k \, x 10^{-6} x$$
$$\left[(1) x \frac{(0.949\hat{x}+0.316\hat{y})}{40} + (-2x) \frac{(0.894\hat{x}+0.447\hat{y})}{20} + (4x) \frac{(0.781\hat{x}+0.625\hat{y})}{41} \right]$$

$$\vec{E} = kx10^{-6} \left[(0.024 - 0.089 + 0.075)\hat{x} + (0.008 - 0.045 + 0.061)\hat{y} \right]$$

$$\vec{E} = kx10^{-6} \left[0.011\hat{x} + 0.024\hat{y} \right] \quad k = 8.99x10^9 \Rightarrow \vec{E} = 98.8\hat{x} + 216\hat{y} \frac{N}{C}$$

(an apparent difference seen by looking at the spreadsheet is due to rounding)

(c) What is the electric potential at the point p?

Note that:

$$V(\vec{r}_p) = \sum_{i=1}^{i=n} k \frac{q_i}{|\vec{r}_p - \vec{r}_i|}$$

$$V = \sum_{i=1}^{3} k \frac{q_i}{|\vec{r}_{ip}|} = k \, x 10^{-6} \left[\frac{1}{\sqrt{40}} + \frac{-2}{\sqrt{20}} + \frac{4}{\sqrt{41}} \right] = 8990 [0.158 - 0.447 + 0.625] = 3021 V$$

(d) How much work did it take to assemble this charge distribution?

$W = W_{12} + W_{13} + W_{23} = q_2 V_1(\vec{r}_{12}) + q_3 V_1(\vec{r}_{13}) + q_3 V_2(\vec{r}_{23})$

$\vec{r}_{12} = \vec{r}_2 - \vec{r}_1 = 2\hat{x} + 0\hat{y} \Rightarrow |\vec{r}_{12}| = 2$

$\vec{r}_{13} = \vec{r}_3 - \vec{r}_1 = 1\hat{x} - 2\hat{y} \Rightarrow |\vec{r}_{13}| = \sqrt{5} = 2.236$

$\vec{r}_{23} = \vec{r}_3 - \vec{r}_2 = -1\hat{x} - 2\hat{y} \Rightarrow |\vec{r}_{23}| = \sqrt{5} = 2.236$

$$W = k\left[\frac{q_1 q_2}{|\vec{r}_{12}|} + \frac{q_1 q_3}{|\vec{r}_{13}|} + \frac{q_2 q_3}{|\vec{r}_{23}|}\right] = 8.99\text{x}10^{-3}[-1 + 1.789 - 3.578] = -0.0251\ \text{J}$$

In a certain region of space, the electric potential is $V = a x^2 + b y^3$. What is the electric field in this region of space?

$$\vec{E} = -\vec{\nabla} V = -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z} = -2ax\,\hat{x} - 3by^2\,\hat{y}$$

(3) Suppose you have a sphere of radius a which has a volume charge density $\rho(\vec{r}) = \rho_0 \frac{r}{a}$. What is the electric field inside and outside the sphere?

Choose a spherical Gaussian surface centered on the sphere. On this surface, E is uniform in direction and magnitude. Thus,

$$\oiint \vec{E}\cdot d\vec{A} = E(4\pi r^2)$$

The enclosed charge is given by:

$$q_{enc} = 4\pi \int_{r=0}^{r} \rho_0 \frac{r}{a} r^2 dr = \frac{4\pi\rho_0}{a}\int_{r=0}^{r} r^3 dr = \frac{4\pi\rho_0}{a}\frac{r^4}{4} = \frac{\pi\rho_0 r^4}{a}$$

If the Gaussian surface is larger than the sphere, then:

$$q_{enc} = 4\pi \int_{r=0}^{a} \rho_0 \frac{r}{a} r^2 dr = \frac{4\pi\rho_0}{a}\frac{a^4}{4} = \pi\rho_0 a^3$$

The electric field is then:

$\vec{E} = \frac{\rho_0 r^2}{4 a \epsilon_0}\hat{r}$ inside the sphere and $\vec{E} = \frac{\rho_0 a^3}{4 r^2 \epsilon_0}\hat{r}$ outside the sphere.

Note that you can find $Q = \frac{\rho_0}{a}\iiint r\, r^2 \sin(\theta)\, dr\, d\theta\, d\phi = \frac{\rho_0}{a}(4\pi)\int r^3 dr = \frac{\rho_0}{a}\frac{(4\pi)}{4}a^4 = \rho_0 \pi a^3$

$$\Rightarrow \rho_0 = \frac{Q}{4\pi} a^3$$

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(4) A parallel plate capacitor has an area A=1000 m^2 and a separation of $1x10^{-3}$m. How much work must be done to charge this capacitor up to a potential of 100V? How much charge is stored on the capacitor. What is the magnitude of the electric field inside the capacitor? If a material with a dielectric constant κ=200 is inserted between the plates of the capacitor (it completely fills the area), what is the capacitance?
If two of these capacitors (with the material) are connected in series, what is the equivalent capacitance. If two of these capacitors (with the material) are connected in parallel, what is the equivalent capacitance?

$$C=\epsilon_0\frac{A}{d}=8.85\,x\,10^{-12}\frac{1000}{1\,x\,10^{-3}}=8.85\,\mu f\approx 10\,\mu f$$

$$|\vec{E}|=\left|\frac{-dV}{dz}\right|=\left|\frac{V}{d}\right|=1x10^{5}\,\frac{V}{m}$$

$$W=\frac{Q^2}{2C}=\frac{1}{2}C\,V^2=\frac{1}{2}\,10x10^{-6}(100)^2=5x10^{-2}\,J$$

$$C=\frac{Q}{V}\Rightarrow Q=CV=10x10^{-6}(100)=1x10^{-3}\,C$$

$$C=\kappa\,C_{geo}=200\,x\,10\,\mu f=2000\,\mu f$$

$$\frac{1}{C_{series}}=\frac{1}{C_1}+\frac{1}{C_2}=\frac{2}{2000}\Rightarrow C_{series}=\frac{1}{\left(\frac{2}{2000}\right)}=\frac{1}{\left(\frac{1}{1000}\right)}=1000\,\mu f$$

$$C_{par}=C_1+C_2=2(2000)=4000\,\mu f$$

(5) Suppose you have two resistors, R_1 and R_2. What is the ratio of the series equivalent resistance to the parallel equivalent resistance if R_1=100Ω and R_2=300Ω?

$$R_{series}=R_1+R_2=400\,\Omega : R_{par}=\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}=\frac{1}{\frac{1}{100}+\frac{1}{300}}=75\,\Omega \Rightarrow \frac{R_{ser}}{R_{par}}=\frac{400}{75}=5.33$$

(6) Calculate the capacitance of a solid conducting sphere of radius a.

$$E\left(4\pi r^2\right)=\frac{Q}{\epsilon_0}\Rightarrow \vec{E}=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r} : V=-\int_\infty^a \vec{E}\cdot d\vec{r}=\frac{-Q}{4\pi\epsilon_0}\int_\infty^a \frac{dr}{r^2}=\frac{Q}{4\pi\epsilon_0}\left[\frac{1}{r}\right]_\infty^a=\frac{Q}{4\pi\epsilon_0 a}=\frac{kQ}{a} : C\equiv\frac{Q}{V}=\frac{a}{k}$$

(7) How much work is required to charge up an ideal coaxial capacitor of inner radius a and outer radius b and length h to a potential difference V?

We had that the potential difference is given by:

$$\Delta V=\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)=\frac{Q}{2\pi\epsilon_0 h}\ln\left(\frac{b}{a}\right)=\frac{Q}{C}\text{ with } C=\frac{2\pi\epsilon_0 h}{\ln\left(\frac{b}{a}\right)}\text{ and } Q=\lambda h$$

Let’s take a small amount of charge across the plates. Initially the capacitor is uncharged. There is no work done for separating the first charge. Now moving the next charge, and successive charges builds up the charge separation and each successive charge requires more work. This:

$$W=0+q\frac{q}{C}+q\frac{2q}{C}+q\frac{3q}{C}+\ldots+q\frac{Nq}{C}=\frac{q^2}{C}\sum_{i=1}^{N} i=\frac{q^2}{C}\frac{\left[N(N+1)\right]}{2}=\frac{qN\left(q(N+1)\right)}{2C}\approx\frac{Q^2}{2C}$$

Or there is always the calculus way: $W=\int_{q=0}^{q=Q}\frac{q}{C}dq=\left[\frac{q^2}{2C}\right]_0^Q=\frac{Q^2}{2C}$

Thus the energy stored is given by: $U=\frac{Q^2}{2C}=\frac{Q^2}{2\left[\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\right]}=\frac{Q^2}{4\pi\epsilon_0 h}\ln\left(\frac{b}{a}\right)$

Now since the actual question asked for this in terms of V and not Q, we also have:

$$Q=CV\Rightarrow U=\frac{1}{2}CV^2=\frac{1}{2}\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}hV^2=\frac{\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}hV^2$$

Again let me point out that the capacitance ought to be independent of both charge and potential difference in a vacuum. If you put a material between the plates of the capacitor, however, anything can happen.