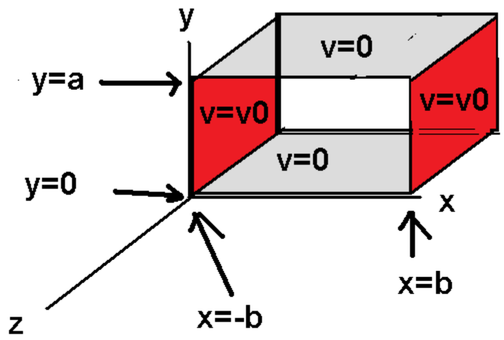
Solutions to Laplace's equation

Separation of variables

Fixed up Example 3.4 (improved 2012)

Additional: Revised order to show after this Example 3.3 and description fixed.



Two grounded planes lie parallel to the xz plane, one at y=0 and the other at y=a. Find the potential in the region between the two planes.

By symmetry, we won't have any z-dependence with our solution. Thus:

$$\vec{\nabla}^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The standard approach to this type of problem is to assume a solution of the form:

$$V(x,y) = X(x)Y(y)$$

We then substitute into Laplace's equation:

$$Y\frac{\partial^2 X}{\partial x^2} + X\frac{\partial^2 Y}{\partial y^2} = 0 \Rightarrow \frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2}$$

Each side of this equation depends only upon an orthogonal variable. Thus, each side must be equal to a constant. Following your author's lead, we then say:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = C_1 \text{ and } \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = C_2 \text{ where } C_1 + C_2 = 0$$

We can immediately provide the connecting condition and write 2 ordinary differential equations that result as:

$$\frac{d^2 X}{dx^2} = k^2 X \text{ and } \frac{d^2 Y}{dy^2} = -k^2 Y$$

It is obvious that the two separation constants must be equal in magnitude and opposite in sign so that they add to zero.

Why choose C_2 negative? The answer is because you want oscillatory solutions between two boundaries held at zero potential. If the other two boundaries were also at zero potential, the solution would not be interesting (zero everywhere).

Having foresight of the problem solution allowed the sign choice here. You ought to be able to do it any way you like … you might have to go into the complex plane for fitting boundary conditions, however.

We can now write the general solutions for X and Y:

$$X(x) = Ae^{kx} + Be^{-kx} : Y(y) = C\sin(ky) + D\cos(ky)$$

you can verify by substitution that this satisfies the two differential equations.

Let's now apply boundary conditions.
In particular I want to look at the Y solutions. At y=0, Y=0. This means that from the general solution, we can say D=0. Our solutions simplify to:

$$X(x)=Ae^{kx}+Be^{-kx} : Y(y)=C\sin(ky)$$

Now look at what happens along x:
Since V(-b)=V(b), we must have that A=B. This gives us:

$$X(x)=Ae^{kx}+be^{-kx}=A\left(e^{kx}+e^{-kx}\right)=2A\cosh(kx)$$

We now write the general solution as:

$$V(x,y)=A\cosh(kx)\sin(ky)$$

where I have combined all constants into one.
Now next look at what happens at y=a: here, V=0. This means:

$$\sin(ka)=0 \Rightarrow ka=n\pi ; n=1,2,3,4,\ldots \Rightarrow k=\frac{n\pi}{a}$$

Now all possible solutions for each k (determined by n) are possible. It is necessary at this point to write the most general solution possible as a linear superposition of all possible solutions since there is no reason that you would pick one particular n over the other at this point. The result is:

$$V(x,y)=\sum_{n=1}^{n=\infty} A_n \cosh\left(n\pi\frac{x}{a}\right)\sin\left(n\pi\frac{y}{a}\right)$$

Now the question is this: how do you determine the various A_n for this expansion. The answer is due to Fourier:
We are left with fitting the boundary at x=b (or x=-b). This is thus: (be careful to not accidentally work the square pipe problem here!)

$$V(x=b,y)=\sum_{n=1}^{n=\infty} A_n \cosh\left(n\pi\frac{b}{a}\right)\sin\left(n\pi\frac{y}{a}\right)=V_0$$

To find the solution to fitting this boundary condition, multiply by

$$\sin\left(m\pi\frac{y}{a}\right)$$

and integrate from 0 to a. Thus:

$$\int_{y=0}^{y=a} V_0 \sin\left(m\pi\frac{y}{a}\right)dy=\int_{y=0}^{y=a} \sin\left(m\pi\frac{y}{a}\right)\sum_{n=1}^{n=\infty} A_n \cosh\left(n\pi\frac{a}{b}\right)\sin\left(n\pi\frac{y}{a}\right)$$

Now what is done is to switch the order of summation and integration. This is fine as long as the functions that are involved provide a complete set. This essentially means that any needed function can be expressed in terms of these functions but I've never been able to correlate this with the mathematicians' terminology. It apparently is not the same as completeness. Physicists just do that so long as they are working with orthogonal functions.

$$\int_{y=0}^{y=a} V_0 \sin\left(m\pi \frac{y}{a}\right) dy = \sum_{n=1}^{n=\infty} A_n \cosh\left(n\pi \frac{b}{a}\right) \int_{y=0}^{y=a} \sin\left(m\pi \frac{y}{a}\right) \sin\left(n\pi \frac{y}{a}\right) dy$$

We might as well now do both integrals.

$$\int_{y=0}^{y=a} V_0 \sin\left(m\pi \frac{y}{a}\right) dy = -\frac{V_0}{\left(\frac{m\pi}{a}\right)} \cos(m\pi) - 1 = -\frac{V_0}{\left(\frac{m\pi}{a}\right)} \begin{bmatrix} 0 & m \in \text{even} \\ 2 & m \in \text{odd} \end{bmatrix}$$

Now we have to hope at this point that something exciting happens to the other side for even m also. Let's see ...

$$\int_{y=0}^{y=a} \sin\left(m\pi \frac{y}{a}\right) \sin\left(n\pi \frac{y}{a}\right) dy = \left[\frac{\sin\left(\frac{y\pi}{2}(m-n)\right)}{2(m-n)\frac{\pi}{a}} - \frac{\sin\left(\frac{y\pi}{2}(m+n)\right)}{2(m+n)\frac{\pi}{a}} \right]_{y=0}^{y=a}$$

$$\Rightarrow \int_{y=0}^{y=a} \sin\left(m\pi \frac{y}{a}\right) \sin\left(n\pi \frac{y}{a}\right) dy = \frac{\sin(\pi(m-n))}{2(m-n)\frac{\pi}{a}} - \frac{\sin(\pi(m+n))}{2(m+n)\frac{\pi}{a}}$$

So long as m and n are unequal integers, this integral is always zero.

Now let's look at what happens when m=n.

$$\int_{y=0}^{y=a} \sin^2\left(m\pi \frac{y}{a}\right) dy = \left[\frac{-\sin\left(2m\pi \frac{y}{a}\right)}{4\left(m\frac{\pi}{a}\right)} \right]_{y=0}^{y=a} = \frac{\sin(2m\pi)}{4\left(m\frac{\pi}{a}\right)} - 0 + 0 = \frac{a}{2}$$

So when m is equal to n, we have a solution, unless n is even, in which case the answer is zero. So I think we're ready: $A_n = 0$ if n is even.

Let's write the series now:

We had:

$$\int_{y=0}^{y=a} V_0 \sin\left(m\pi \frac{y}{a}\right) dy = \sum_{n=1}^{n=\infty} A_n \cosh\left(n\pi \frac{b}{a}\right) \int_{y=0}^{y=a} \sin\left(m\pi \frac{y}{a}\right) \sin\left(n\pi \frac{y}{a}\right) dy$$

Now, when we did

$$\int_{y=0}^{y=a} \sin\left(m\pi \frac{y}{a}\right) \sin\left(n\pi \frac{y}{a}\right) dy$$

the result was that we have zero unless m=n.

This **means that the sum collapses** down to **one single term**, namely that one for which m=n.

So we now have:

$$\int_{y=0}^{y=a} V_0 \sin\left(m\pi \frac{y}{a}\right) dy = A_m \cosh\left(m\pi \frac{b}{a}\right)\frac{a}{2}$$

But when we evaluated the other integral, we found that if m is even, the integral vanishes. This means we only have the odd m present. This then gives:

$$A_m \cosh\left(m\pi \frac{b}{a}\right)\frac{a}{2} = \begin{bmatrix} 0 & m\in\text{even} \\ \dfrac{2V_0 a}{m\pi} & m\in\text{odd} \end{bmatrix}$$

The solution for the A_m is then:

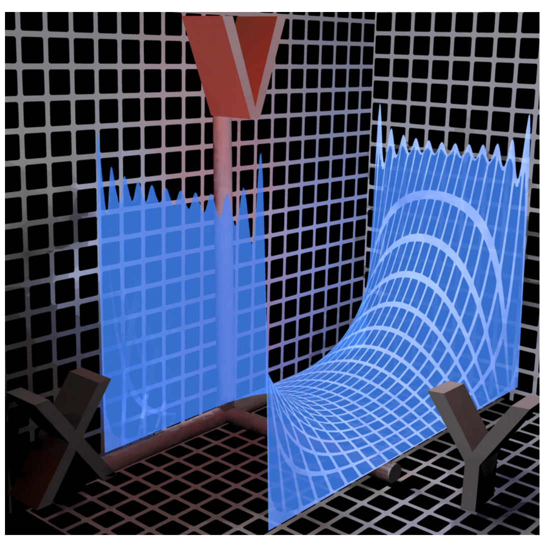
$$A_m = \frac{2V_0 a}{m\pi \cosh\left(m\pi \frac{b}{a}\right)}\frac{2}{a} = \frac{4V_0}{m\pi \cosh\left(m\pi \frac{b}{a}\right)}$$

In terms of n, this reads:

$$A_n = \frac{4V_0}{n\pi \cosh\left(n\pi \frac{b}{a}\right)}$$

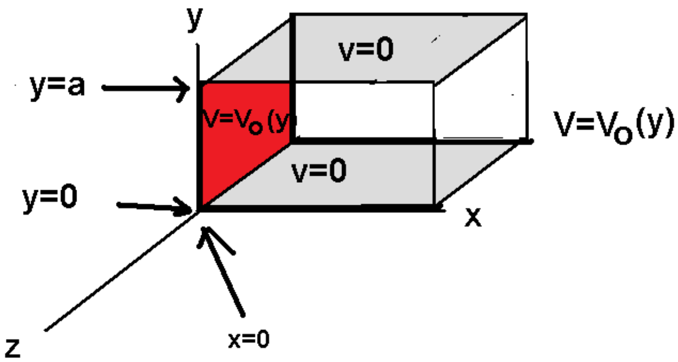
And I can now write the solution:

$$V(x,y) = \sum_{\substack{n=1 \\ n\in\text{odd}}}^{n=\infty} \frac{4V_0}{n\pi \cosh\left(n\pi \frac{b}{a}\right)} \cosh\left(n\pi \frac{x}{a}\right) \sin\left(n\pi \frac{y}{a}\right) = \frac{4V_0}{\pi} \sum_{\substack{n=1 \\ n\in\text{odd}}}^{n=\infty} \frac{1}{n} \frac{\cosh\left(n\pi \frac{x}{a}\right)}{\cos\left(n\pi \frac{b}{a}\right)} \sin\left(n\pi \frac{y}{a}\right)$$



The figure 3.21 by the author has only a very few terms present.

Now look at example 3.4. It will be easier since the dimension is lower (maybe).
The BC are shown below. We shall assume the BC at infinity is V=0.
In fact, after the last problem this will go very fast.



The Laplacian is for this problem:

$$\vec{\nabla}^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

The general solution here is sinusoidal in Y and exponential in x.
The general solution is then:

$$V(x,y) = \left[C e^{kx} + D e^{-kx}\right]\left[A \sin(ky) + B \cos(ky)\right]$$

Because there is no z-dependence (by the model).

Now A=0 to fit the BC at y=0
Further, ka=nπ,n=1,2,3,... to fit the BC at y=a
Next: at infinity the potential is zero so C=0.
We thus have till now, for a particular value of n,

$$V(x,y)_n = A_n \sin\left(n\pi \frac{y}{a}\right) e^{-kx}$$

where I have collapsed the constants into A.
Now in general we need to take the linear superposition in order to get the most general solution. This is then:

$$V(x,y) = \sum_{n=1}^{n=\infty} A_n \sin\left(n\pi \frac{y}{a}\right) e^{-kx}$$

Next, we need to evaluate the condition at x=0. This is then:

$$V(x=0,y) = \sum_{n=1}^{n=\infty} A_n \sin\left(n\pi \frac{y}{a}\right) = V_0(y)$$

Now we do Fourier's trick:

$$\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy=\sum_{n=1}^{n=\infty} A_n\int_{y=0}^{y=a}\sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)dy=A_m\frac{a}{2}$$

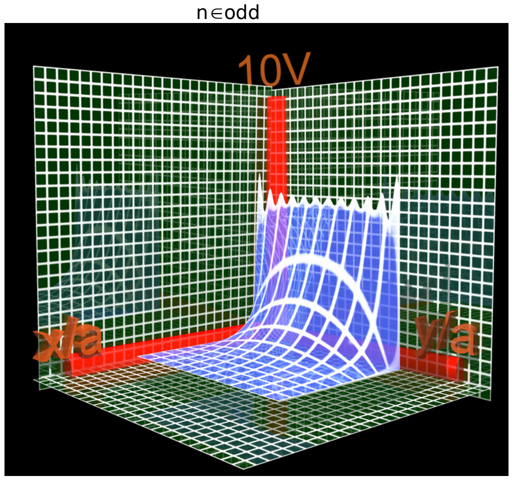
$$\Rightarrow A_m=\frac{2}{a}\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy$$

Now suppose $V_0(y)$ is a constant, namely V_0.

$$A_m=\frac{2}{a}\int_{y=0}^{y=a}\sin\left(m\pi\frac{y}{a}\right)dy=\left[\frac{-2}{a}V_0\frac{1}{m\frac{\pi}{a}}\cos\left(m\pi\frac{y}{a}\right)\right]_{y=0}^{y=a}=\frac{-2V_0}{m\pi}\cos(m\pi)-1=\begin{bmatrix}\frac{4V_0}{m\pi} & m\in odd\\ 0 & m\in even\end{bmatrix}$$

So the general solution is now given by:

$$V(x,y)=\frac{4V_0}{\pi}\sum_{\substack{n=1\\ n\in odd}}^{n=\infty}\frac{1}{n}\sin\left(n\pi\frac{y}{a}\right)e^{-n\pi\frac{x}{a}}$$



Note that the sketch that your author gave (Figure 3.18) is not really a plot of the potential. In this case, I believe your author only used the first term of the series expansion. The true potential would not show the rounded edges.

Here is my sketch of the potential with terms up to n=23:

According to your author, this series can be summed to give:

$$V(x,y)=\frac{2V_0}{\pi}\tan^{-1}\left(\frac{\sin\left(\pi\frac{y}{a}\right)}{\sinh\left(\pi\frac{x}{a}\right)}\right)$$

For the previous problem, without the charge between the planes, we have:

$$V\left(\vec{r}_p\right)=kq\left[\sum_{m=1}^{\infty}\frac{(-1)^m}{\sqrt{x_p^2+y_p^2+\left(z_p-2md\right)^2}}+\sum_{m=1}^{\infty}\frac{(-1)^m}{\sqrt{x_p^2+y_p^2+\left(z_p+2md\right)^2}}\right]$$

I suppose we'll need to sum this infinite series. (we don't, however, need to do this to prove convergence). At least we can say the series does converge. If I ever do sum this thing, I'll let you know. Here is something that I want to point out, though … this series converges to zero as x and y get large. Z won't get large because it stays between the two planes. The point here is that it won't provide enough to subtract off any infinity from the original problem. Thus, we're left with a problem … our original problem is quite likely going to diverge at infinity.

My way around this is the following: I'm going to modify the original problem to specifically state that there is a grounded plane at infinity terminating the system.
This is, however, not the original problem and I've almost come to the conclusion that the original problem as solved in the example might be in error.

Let me explicitly show you my difficulty with the bc ,
Suppose a long wire has a uniform charge density □.
Find the electric field and the potential.
The solution is to assume a cylindrical Gaussian cylinder:

$$\Phi_E=\oiint_{surface}\vec{E}\cdot d\vec{A}=\frac{q_{enc}}{\epsilon_0}\Rightarrow E\left(2\pi sh\right)=\frac{\lambda h}{\epsilon_0}\Rightarrow\vec{E}=\frac{\lambda}{2\pi\epsilon_0}\hat{s}$$

We can find the potential difference between two points:

$$\Delta V=-\int_a^b\vec{E}\cdot d\vec{s}=\frac{-\lambda}{2\pi\epsilon_0}\int_a^b\frac{ds}{s}=\frac{-\lambda}{2}\epsilon_0\ln\left(\frac{b}{a}\right)$$

Now if you reference a to be infinity, you have the potential but this clearly shows that the potential would diverge. Our strip here is not really all that different from the long straight wire, so we'll expect that it's going to diverge at infinity also.

In a real problem, however, the strip won't go to infinity and in such a situation, at a large enough distance from the strip, it will look like a point charge which then predicts the potential vanishes at 1/r.

The rest of the boundary conditions are easy enough to apply.

According to the BC at infinity, we then have A=0.
Our x-solution then becomes:

$$X(x)=Be^{-kx}$$

We thus have as our potential solution:

$$V(\vec{r}) = Be^{-kx}(C\sin(ky) + D\cos(ky))$$

We can now eliminate D since at y=0, we need the potential to vanish. Thus:

$$V(\vec{r}) = BCe^{-kx}\sin(ky) = Be^{-kx}\sin(ky)$$

(I've set C equal to 1)

Now we apply the condition at y=a:

At y=a, we need V=0. Thus:

$$\sin(ka) = 0 \Rightarrow ka = n\pi \Rightarrow k = \frac{n\pi}{a}; n = 1,2,3,\ldots$$

We exclude n=0 since that won't allow a solution in other regions. We disallow the negative n solutions because of the fact that I could group the solutions in such a way so as to produce zero always.

What we then have is an infinite set of solutions to the potential given by:

$$V(\vec{r}_p) = B_n e^{-n\pi\frac{x}{a}}\sin\left(n\pi\frac{y}{a}\right); n = 1,2,3,\ldots$$

We need to find out what the solution for our present problem is. The important point here is that solutions to Laplace's equation show linear superposition. This means:

If V_1 and V_2 are solutions then $\alpha_1 V_1 + \alpha_2 V_2$ is also a solution.

We use this to fit the final boundary condition.

At x=0, for y between the planes, we have:

$$V_0(y) = \sum_{n=1}^{n=\infty} B_n \sin\left(n\pi\frac{y}{a}\right)$$

Your author presents an excellent discussion of what to do now on page 130. You'll want to read it closely. Here it is:

$$V(0,y) = V_0(y) = \sum_{n=1}^{n=\infty} B_n \sin\left(n\pi\frac{y}{a}\right)$$

Let's multiply this and integrate over the interval in order to find the Bn values:

(This is called Fourier's Trick):

$$V_0(y)\sin\left(m\pi\frac{y}{a}\right) = \sum_{n=1}^{n=\infty} B_n \sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)$$

$$\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy = \int_{y=0}^{y=a}\sum_{n=1}^{n=\infty} B_n \sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)dy$$

Now, we can reverse the order of summation and integration if the series is complete. Normally, we don't need to worry about completeness … if it's proven for the functions we use, we don't need to prove it every time we use it. Thus, we'll assume completeness here. This gives:

$$\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy=\sum_{n=1}^{n=\infty}\int_{y=0}^{y=a} B_n\sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)dy$$

We can simplify this one step further:

$$\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy=\sum_{n=1}^{n=\infty} B_n\int_{y=0}^{y=a}\sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)dy$$

Let's evaluate this integral.
We can scale it:

$$\text{let}\quad u=\pi\frac{y}{a}\Rightarrow dy=\frac{a\,du}{\pi}:y=0\Rightarrow u=0:y=a\Rightarrow u=\pi$$

$$\int_{y=0}^{y=a}\sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)dy=\frac{a}{\pi}\int_{u=0}^{u=\pi}\sin(nu)\sin(mu)\,du$$

If you go to the alpha site, and enter integrate sin(nx)*sin(mx) dx
the result is:

$$\frac{n\sin(mx)\cos(nx)-m\cos(mx)\sin(nx)}{m^2-n^2};m\neq n$$

$$\left.\frac{n\sin(mu)\cos(nu)-m\cos(mu)\sin(nu)}{m^2-n^2}\right|_{u=0}^{u=\pi}=\frac{n\sin(m\pi)\cos(n\pi)-m\cos(m\pi)\sin(n\pi)}{m^2-n^2}=0$$

When n=m, enter integrate sin(nx)*sin(nx) dx
the result is

$$\frac{x}{2}-\sin\frac{(2nx)}{4n}$$

$$\left.\frac{u}{2}-\frac{\sin(2nu)}{4n}\right|_{u=0}^{u=\pi}=\frac{\pi}{2}$$

$$\int_{y=0}^{y=a}\sin\left(n\pi\frac{y}{a}\right)\sin\left(m\pi\frac{y}{a}\right)dy=\begin{bmatrix}\frac{a}{2} & m=n\\ 0 & m\neq n\end{bmatrix}$$

So we are left with this:

$$\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy=\frac{B_m a}{2}\Rightarrow B_m=\frac{2}{a}\int_{y=0}^{y=a} V_0(y)\sin\left(m\pi\frac{y}{a}\right)dy$$

This picks out a particular situation then to give us the coefficients:
Firstly, we represent both of these cases as:

The rest of the story then involves solving this integral for whatever potential you specify and then writing down the general solution. Let me show you a simple example. Suppose

$$V_0(y)=V_0$$

We then have:

$$B_n=\frac{2V_0}{a}\int_{y=0}^{y=a}\sin\left(n\pi\frac{y}{a}\right)dy$$

Where for simplicity I've renamed m as n.

We can evaluate this integral.

$$w\equiv n\pi\frac{y}{a}\Rightarrow dy=\frac{a}{n\pi}dw\,;y=0\Rightarrow w=0:y=a\Rightarrow w=n\pi$$

$$B_n=\frac{2V_0}{a}\frac{a}{n\pi}\int_{w=0}^{w=n\pi}\sin(w)dw=\frac{-2V_0}{n\pi}[\cos(w)]_0^{n\pi}=\frac{4V_0}{n\pi}\begin{bmatrix}0 & n\in even\\ 1 & n\in odd\end{bmatrix}$$

You can now write the general solution:

$$V(x,y)=4\frac{V_0}{\pi}\sum_{n=1,3,4,...}^{\infty}\frac{1}{n}\sin\left(n\pi\frac{y}{a}\right)e^{-n\pi\frac{x}{a}}$$

According to your author, this series can be summed to give:

$$V(x,y)=\frac{2V_0}{\pi}\tan^{-1}\frac{\sin\left(\pi\frac{y}{a}\right)}{\sinh\left(\pi\frac{x}{a}\right)}$$

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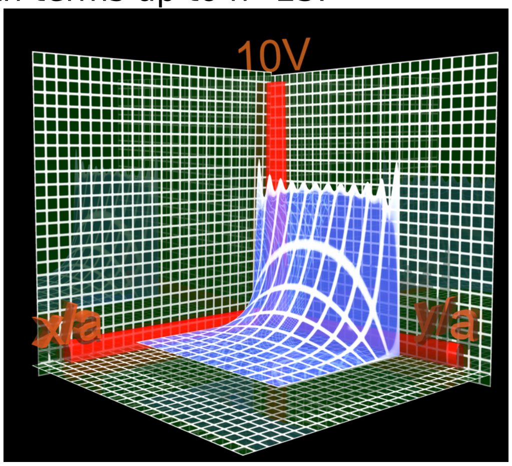

You'll find a tif version of this on the website. It's quite big. The peaks at x=0 represent the failure to include an infinite number of terms in the Fourier Series.

This method of solution hinges on two important things:

(1) Completeness:

According to your author, a set of functions $f_n(y)$ is complete if any other function $f(y)$ can be expressed as a linear combination of this set of functions. That is:

$$f(y)=\sum_{n=1}^{\infty} f_n(y)$$

The other property is

(2) Orthogonality:

A set of functions is said to be orthogonal on an interval if:

$$\int_{y=0}^{y=a} f_n(y) f_m(y)\,dy=0 \text{ for } n \neq m \ .$$

Notice the small error in equation 3.39.