The Parallel-plate diode

(derivation of the Child-Langmuir Law)
See Marion, Classical Electromagnetic Radiation, 2nd Edition, Page 94.
Poisson's Equation:

$$\nabla^2 V = \frac{-\rho}{\epsilon_0}$$

We need this because the space is not going to be charge free.
Assume electrons are released from the cathode at zero potential with zero velocity.
Then, the increase in kinetic energy of the electrons is due to the potential change:

$$\frac{1}{2} m u^2 = e V$$

The current density , $J = I/A$ in terms of velocity and charge density is:

$$J = |\rho u|$$

where the absolute value eliminates the possibility of problems due to a negative sign.
Since electrons are the current we remove the negative sign. We then have:

$$\frac{d^2 V}{dx^2} = \frac{J}{\epsilon_0 u} = \frac{J}{\epsilon_0 \sqrt{\frac{2eV}{m}}}$$

To integrate this equation, multiply it by dV/dx:

$$\frac{dV}{dx}\frac{d^2 V}{dx^2} = \frac{1}{2}\frac{d}{dx}\left(\frac{dV}{dx}\right)^2 = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\, V^{-1/2}\frac{dV}{dx} = \frac{1}{2}\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\frac{dV^{1/2}}{dx}$$

We thus have: ... this could be cleaner ...

$$\frac{d}{dx}\left(\frac{dV}{dx}\right)^2 = \frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\frac{dV^{1/2}}{dx} = \frac{d}{dx}\left[\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\, V^{1/2}\right]$$

This is seen to become:

$$\left(\frac{dV}{dx}\right)^2 = \left[\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\, V^{1/2}\right]$$

by direct integration.
The constant of integration vanished because the electrons are released with zero velocity. (V=0 and dV/dx=0 at x=0).

$$\left(\frac{dV}{dx}\right) = \left[\frac{J}{\epsilon_0}\sqrt{\frac{m}{2e}}\right]^{1/2} V^{1/4} = \sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} V^{1/4}$$

$$\int_{V=0}^{V=V} V^{-1/4}\, dV = \frac{V^{3/4}}{3/4} : \int_{x=d}^{x=0} \sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} dx = -\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4} d \Rightarrow V^{3/4} = \frac{-4}{3}\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4}$$

We then have:

$$\left(\frac{dV}{V^{1/4}}\right)=\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4}dx\Rightarrow V^{3/4}=\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4}x$$

At x=d, we will let $V=V_0$. We can then solve for the current density:

$$V_0^{3/4}=\sqrt{\frac{J}{\epsilon_0}}\left(\frac{m}{2e}\right)^{1/4}d\Rightarrow\epsilon_0\frac{V_0^{3/2}}{d^2}\sqrt{\frac{2e}{m}}=J\ .$$

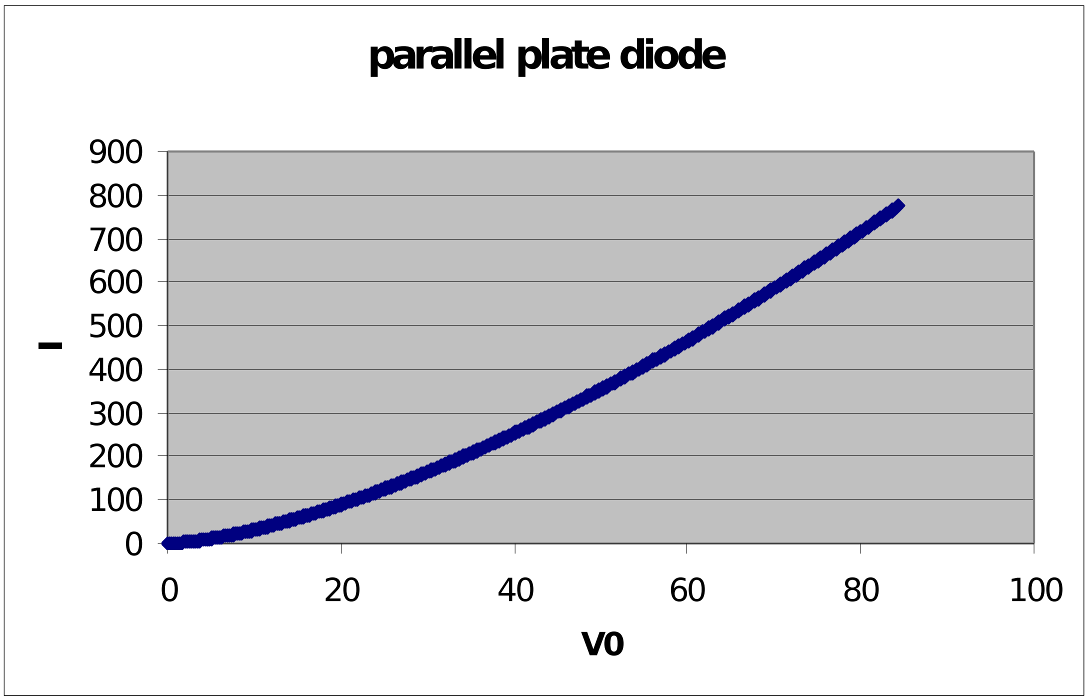
The observation that $J\propto V_0^{3/2}$ is known as the Child-Langmuir Law.

In this case, it also clearly shows that the parallel plate diode is not an Ohmatic device.

We can, incidentally, rewrite this: multiply both sides by A to obtain:

$$JA=I=\epsilon_0\frac{A}{d}\frac{V_0^{3/2}}{d}\sqrt{\frac{2e}{m}}=C_{geo}\frac{V_0^{3/2}}{d}\sqrt{\frac{2e}{m}}$$

where I have used the capacitance of a parallel plate capacitor (not necessarily appropriate).Increasing the capacitance will increase the current



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