3.1 Laplace's Equation

I have previously shown you two differential equations for the potential:

$$\vec{\nabla}^2 V = -\frac{\rho}{\epsilon_0}$$

and in charge-free regions of space, we have:

$$\vec{\nabla}^2 V = 0$$

In Cartesian Coordinates, we have:

$$\vec{\nabla}^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \Rightarrow \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

One dimension.

In the one-dimensional case, Laplace's equation has particularly simple solutions:

$$\frac{d^2 V}{dx^2} = 0 \Rightarrow V(x) = mx + b$$

You can see that the two undetermined constants can be used to fix the boundary conditions. For example, suppose that V=4 at x=1 and V=0 at x=5. Then m=-1 and b=5 so that V=-x+5.

Your author goes through great lengths to make some points about this equation in one dimension.

(1) V(x) is the average of V(x+a) and V(x-a). For any a:

$$V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$$

It does give some insight into numerical solution techniques to realize that the value of the potential at a point is the average of the potentials on either side of the point.

You should also note, although I'm not going to prove it now, that Laplace's equation has no local maxima nor minima.

Two dimensions

In 2 dimensions, Laplace's equation becomes:

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}=0$$

This is going to give us a system of much more exciting solutions to V than the one dimensional case. These solutions are called "Harmonic functions in two dimensions."

Your author states that the potential at a point is still the average:

$$V(x,y)=\frac{1}{2\pi R}\oint_{circle} V dl$$

of a circle of potential centered on that point. This observation is used to provide an iterative solution to Laplace's equation called the "method of relaxation.

As was the case for 1-D, these solutions have no local maxima or minima and all extrema must occur at the boundaries. Your author points out that these solutions of the most featureless you can imagine, devoid of hills and valleys.

Three dimensions

Laplace's equation in 3 dimensions

$$\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0$$

in rectangular Cartesian coordinates.

Point 1:
The value of V at point r is the average value of V over a spherical surface of radius R centered at r:

$$v(\vec{r})=\frac{1}{4\pi R^2}\oint\!\!\oint_{sphere} V\,da$$

Look at the divergence and curl notes. We had:

We'll follow your author.

Suppose you have a charge q located at $\vec{r}_i=x_i\hat{x}+y_i\hat{y}+z_i\hat{z}$.
Then, at a distance r from the charge, we have:

$$V(\vec{r}_p)=\frac{q_i}{4\pi\epsilon_0|\vec{r}_p-\vec{r}_i|}$$

We can find the vector difference by:

$$\vec{r}_p-\vec{r}_i=(x_p-x_i)\hat{x}+(y_p-y_i)\hat{y}+(z_p-z_i)\hat{z}$$

Thus, we have:

$$|\vec{r}_p-\vec{r}_i|=\sqrt{(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2}$$

To find the average potential over a sphere, we then need to evaluate:

$$V(\vec{r}) = \frac{1}{4\pi R^2} \oiint_{sphere} V da = \frac{1}{4\pi R^2} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{q R^2 \sin(\theta) d\theta d\phi}{4\pi\epsilon_0 \sqrt{(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2}}$$

Now, we're going to have to convert some of these coordinates, one way or another. I'm going to do it by brute force (it's actually always done this way)

$$V(\vec{r}) = \frac{q}{16\pi^2 \epsilon_0} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\sin(\theta) d\theta d\phi}{\sqrt{x_p^2 - 2 x_p x_i + x_i^2 + y_p^2 - 2 y_p y_i + y_i^2 + z_p^2 - 2 z_p z_i + z_i^2}}$$

You need to remember here just what we're integrating over. We are integrating over the points in space (not the charges, as we normally do). This means that the variables of interest are the "p" variables. Thus, we're going to want to convert those variables to spherical coordinates.

Without loss of generality that we can permit the charge to have coordinates given by: $(0,0,z_i)$.

Then:

$$V(\vec{r}) = \frac{q}{16\pi^2 \epsilon_0} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\sin(\theta) d\theta d\phi}{\sqrt{x_p^2 + y_p^2 + z_p^2 - 2 z_p z_i + z_i^2}}$$

From the useful sheet: $x = r\sin(\theta)\cos(\phi) : y = r\sin(\theta)\sin(\phi) : z = r\cos(\theta)$

Then

$$V(\vec{r}) = \frac{q}{16\pi^2 \epsilon_0} \int_{\theta=0}^{\theta=\pi} \int_{\phi=0}^{\phi=2\pi} \frac{\sin(\theta) d\theta d\phi}{\sqrt{R^2 - 2 z_i R\cos(\theta) + z_i^2}}$$

$$V(\vec{r}) = \frac{q}{8\pi\epsilon_0} \int_{\theta=0}^{\theta=\pi} \frac{\sin(\theta) d\theta}{\sqrt{R^2 - 2 z_i R\cos(\theta) + z_i^2}}$$

Now, we're going to do the other integral.

At the alpha site, enter: integrate sin(x) / ((a -b*cos(x))^(1/2))

The result is:

$$\int \frac{\sin(x)}{\sqrt{a - b\cos(x)}} dx = \frac{2\sqrt{a - b\cos(x)}}{b}$$

$$\Rightarrow \int_{\theta=0}^{\theta=\pi} \frac{\sin(\theta)}{\sqrt{a - b\cos(\theta)}} d\theta = \frac{2\sqrt{a+b}}{b} - \frac{2\sqrt{a-b}}{b} = \frac{2}{b}\left(\sqrt{a+b} - \sqrt{a-b}\right)$$

$$\Rightarrow V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \frac{\sqrt{R^2 + z_i^2 + 2 z_i R} - \sqrt{R^2 + z_i^2 - 2 z_i R}}{2 z_i R} = \frac{1}{4\pi\epsilon_0} \frac{\left[\sqrt{(R + z_i)^2} - \sqrt{(R - z_i^2)}\right]}{2 z_i R}$$

Case 1: $z_i > R$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\left[R + z_i - z_i + R\right]}{2 z_i R} = \frac{1}{4\pi\epsilon_0 z_i}$$

Case 2: $z_i = R$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\frac{[z_i + z_i - z_i + z_i]}{2 z_i z_i} = \frac{1}{4\pi\epsilon_0 z_i}$$

Case 3: $z_i < R$

Here, Laplace's equation does not hold.

In this situation, you find the average potential on a sphere by superposition:

$$V_{average} = V_{center} + \frac{Q}{4\pi\epsilon_0 R}$$

Where V_{center} is the potential at the center due to all external charges and Q is the total enclosed charge. (look at problem 3.1)

Earnshaw’s Theorem:

In a charge free region of space, where Laplace’s equation holds we can have no local maxima or minima. Why? It’s because we could draw a sphere around the maxima, without having the maxima touch the surface of the sphere. All points on the sphere would have a lower value than the maxima (or a higher value than the minima). Thus, the theorem above would be violated.

How does this show Earnshaw’s theorem?

In order for a charge to be in stable equilibrium, it must be in a region of space where there is a local minimima in the potential. These local minima do not occur as a result of Laplace’s equation. Thus, you can not hold a particle in stable equilibrium purely as a result of electrostatic forces.

Problem 3.3: Find the general solution to Laplace’s Equation in Spherical Coordinates for V(r) and in cylindrical coordinates for V(s) only.

Solution:

Spherical coordinates, 1 dimensional

$$\vec{\nabla}^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) = 0 \Rightarrow \frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \Rightarrow r^2 \frac{dV}{dr} = C \Rightarrow \frac{dV}{dr} = \frac{C}{r^2} \Rightarrow V = V_0 - \frac{C}{r}$$

Cylindrical coordinates, 1 dimensional

$$\vec{\nabla}^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s \frac{\partial V}{\partial s}\right) = 0 \Rightarrow \frac{d}{ds}\left(s \frac{dV}{ds}\right) = 0 \Rightarrow s \frac{dV}{ds} = C \Rightarrow \frac{dV}{ds} = \frac{C}{s} \Rightarrow V = V_0 + \ln\left(s^C\right)$$

3.1.5 Uniqueness Theorems

I'll just quote from your author here without proof so we can move on.

First Uniqueness Theorem

The potential in some volume V is uniquely determined if V is specified on some bounding surface S.

corollary: The potential in a volume V is uniquely determined if V is specified on some bounding surface and the charge density is specified throughout the volume.

Second Uniqueness Theorem

In a volume V surrounded by conductors and containing a specified charge density, ρ , the electric field is uniquely determined if the total charge on each conductor is given

I strongly recommend reading closely page 120, 121.

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