The theory of vector fields : the Helmholtz theorem

"to what extent is a vector field determined by its divergence and curl?"

Suppose:

$\vec{\nabla}\cdot\vec{V}=D$ and $\vec{\nabla}\times\vec{V}=\vec{C}$ with $\vec{\nabla}\cdot\vec{C}=0$

is it possible to determine $\vec{V}$ uniquely?

The answer provided by the Helmholtz theorem is yes, provided that the fields satisfy the boundary condition that they vanish at infinity. For all physical problems involving electric and magnetic fields this is true but, for infinite lines (of charges , currents) or infinite planes (of charges , currents) (as examples), since the fields do not necessarily vanish at infinity, your author states that you need to use symmetry arguments in order to provide boundary conditions.

This then leads up to the important concept of potentials:

Note: V is called the "scalar potential."

Irrotational fields: if $\vec{\nabla}\times\vec{F}=\vec{0}\Leftrightarrow\vec{F}=-\vec{\nabla}V$

If $\vec{F}$ satisfies one of these conditions, it satisfies all of them:

(a) $\vec{\nabla}\times\vec{F}=\vec{0}$ everywhere

(b) $\oint_a^b \vec{F}\cdot d\vec{L}$ does not depend upon the path between two endpoints a and b.

(c) $\oint_{\text{closed path}} \vec{F}\cdot d\vec{L}=0$ for any correctly closed path

(d) $\vec{F}=-\vec{\nabla}V$ for some scalar function V. (negative is physics convention)

More generally: $\vec{F}=-\vec{\nabla}V+\vec{\nabla}\times\vec{A}$ ($\vec{F}$ is a irrotational vector field)

The scalar potential V is not uniquely determined: any constant (or, in fact any curl of any vector function) can be added to it. This is because

$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})\equiv 0$ (problem 1.26)

This will lead later to being able to choose a particular gauge.

Note: $\vec{A}$ is called the "vector potential."

rotational fields: $\vec{\nabla}\cdot\vec{F}=0\Leftrightarrow\vec{F}=\vec{\nabla}\times\vec{A}$.

If $\vec{F}$ satisfies one of these conditions, it satisfies all of them:

(a) $\vec{\nabla}\cdot\vec{F}=0$ everywhere

(b) $\oint\vec{F}\cdot d\vec{A}$ is independent of the surface, for any given (finite) boundary line

(c) $\oint\vec{F}\cdot d\vec{A}=0$ for any close surface

(d) $\vec{F}=-\vec{\nabla}V+\vec{\nabla}\times\vec{A}$ ($\vec{F}$ is a solenoid vector field)

The vector potential $\vec{A}$ is not uniquely determined: any constant (or, in fact any divergence of any scalar function) can be added to it. This is because

$\vec{\nabla}\times(\vec{\nabla}V)\equiv 0$ (problem 1.27)

This will lead later to being able to choose a particular gauge.