2016	Physics **250**: Worksheet 04	Name _______________

Concepts: Electrostatic Work, Electrostatic Potential, superposition, Conservative Fields, Electrostatic Potential difference, capacitance, $\vec{E}=-\frac{\Delta V}{\Delta x}\hat{x}$.

(1) Find the work required to bring a charge q_p from infinity to a location $\vec{r}_p$ in the presence of an arbitrary charge distribution which produces an electrostatic potential V. Thus, use this to introduce the electrostatic potential.

(2) Find the electrostatic potential due to a conducting sphere of radius a (located at the origin) at a point $\vec{r}_p$ from the center of the sphere if the sphere has a total charge Q.

(3) Determine the work required to assemble 3 charges given by:

(#,q,x,y,z)=(1,3μC,1,1,1),(2,6μC,2,2,2),(3,-9μC,3,3,3).

(4) The potential difference is defined as the difference in electrostatic potential between two points in space. It can also be defined as the negative work per unit charge to move a charge between two points in space.
Determine the potential difference between two plates of a ideal parallel plate capacitor with plates of area A and separation d which has a charge Q on one plate and –Q on the other plate.

(5) A parallel plate capacitor of capacitance C=6μf is charged to a potential difference of 200 Volts. How much energy is stored in the capacitor? If the plate separation is 0.0001m, what is the energy density of the capacitor? Then, what is the magnitude of the electric field inside the capacitor?

Electrostatic Potential due to a Point Charge

Consider a charge located at the origin. The field from this charge is given by:

$$\vec{E}_p = k\frac{q_1}{|\vec{r}_p - \vec{r}_1|^2}\hat{r}_{1p} = k\frac{q_1}{|\vec{r}_p|^2}\hat{r}_p$$

Now, let's bring in a second charge (q_p) from infinity and ask how much work is required to do this. The force which must be overcome is the electrostatic force and thus,

$$\vec{F}_{p1} = k\frac{q_1 q_p}{|\vec{r}_p - \vec{r}_i|^2}\hat{r}_{1p}$$

From our definition of work, we can determine this result:

$$W_{1p} = \sum_{\delta \vec{r}_p} \vec{F}_{p1} \cdot \vec{r}_p$$

The problem here is that the force is varying as you get charge p closer and closer to charge 1 (that is why I needed to include the sum sign). To answer the ultimate question of how much work is required to bring charge p to within a distance $|\vec{r}_p - \vec{r}_i|$, you need to add up small changes in the positions.

Non-calculus:

For non-calculus people, the result is:

$$W_{1p} = \frac{k q_1 q_2}{|\vec{r}_p|}$$

Calculus:

$$W_{1p} = \int_{\infty}^{|\vec{r}_p - \vec{r}_i|} k\frac{q_1 q_p}{|\vec{r}_p - \vec{r}_i|^2}\hat{r}_{1p} \cdot d\vec{r}_{1p}$$

where

$$\vec{r}_{ip} = \vec{r}_p - \vec{r}_i$$

The easiest thing to do now is to set charge 1 at the origin. Thus,

$$W_{1p} = \int_{\infty}^{|\vec{r}_p|} k\frac{q_1 q_p}{\vec{r}_p^{\,2}}\hat{r}_p \cdot d\vec{r}_p$$

$$W_{1p} = k q_1 q_p \int_{\infty}^{|\vec{r}_p|} \frac{dr_p}{r_p^{\,2}}\hat{r} = \left(\frac{k q_1 q_p}{r_p}\right)\Bigg|_{\infty}^{|\vec{r}_p|} = \frac{k q_1 q_p}{|\vec{r}_p|}$$

We like to side-step this whole work thing though since the work really depends upon the size of the charge that you are bringing up from infinity. Instead, we define the **electrostatic potential** as

$$V \equiv \frac{W}{q_p}$$

For the case of a single point charge q_i located at $\vec{r}_i$, the electrostatic potential at a point in space designated by $\vec{r}_p$ is given by

$$V = k\frac{q_i}{\left|\vec{r}_p - \vec{r}_i\right|}$$

This expression is very important

Furthermore, since in reality all charges are discrete, it can be argued that one obtains all other potentials by a direct superposition of individual potentials due to point charges. Often, since this is not a vector quantity, you might find it easier to calculate the potential and from there calculate the electric field. However, unless you're a calculus student or you have a very simple situation (such as a parallel plate capacitor), you may find this task to be difficult.

Here is a nice example … calculate the potential along the symmetry axis of the ring of charge which we had in an earlier problem on worksheet 2.

Solution:

In this case, each of the charges is the same distance from the point along the symmetry axis. Thus, if the ring is in the x-y plane and the symmetry axis is along the z-direction, we have:

$$V\left(z_p\right) = \frac{kQ}{\sqrt{z_p^2 + a^2}}$$

where the radius of the ring is a and the ring has a total charge Q. Now, you might ask, what good is this potential? Here is the answer (in the form of a question): How much work is required to bring a positive test charge q_p from infinity to a distance z_p from a positively charged ring of radius a with a total charge Q?

The answer is pretty straight forward now:

$$W = q_p V = \frac{kQq_p}{\sqrt{z_p^2 + a^2}}$$

The nice thing about electrostatic potentials is this: they are conservative. This means that no matter what path you take from infinity to the desired location, if you can calculate the potential at two reference points, the work in going from the initial position to the final position is always the same (independent of path).

For non-calculus students, the result for the electric field along the symmetry axis of the ring is

$$\vec{E} = \frac{kQz_p}{\left(z_p^2 + a^2\right)^{3/2}}\hat{z} \quad .$$

Now for calculus students, we can also use the potential to find the electric field easily. Here are some details: The electro**static** field is a **conservative vector field**. This means several important things but for now, this means that there exists a scalar potential such that:

$$\vec{E} = -\vec{\nabla}_p V$$

where the "p" subscript indicated the gradient is with respect to the p coordinates, not the charge coordinates.

If E is uniform in the x-direction, this connection is particularly easy to work with:

$$\vec{E} = -\frac{\Delta V}{\Delta x}\hat{x}$$

This is the most complicated version of this that the non-calculus students can work with. By the way, if you need to review the differential operators, look at the link to the "useful page" which is on the class website.

This allows us to go both ways: knowledge of E allows us to find V and knowledge of V allows us to find E. Which one you actually calculate depends upon which you find easier to produce. Again, let me emphasize the word electrostatic here.

Now return to the ring problem:

$$\vec{E} = -\vec{\nabla}_p V = \frac{-dV}{dz}\hat{z} = -kQ\frac{d\left(z_p^2+a^2\right)^{-1/2}}{dz_p}\hat{z} = -kQ\left[\frac{-1}{2}\left(2z_p\right)\left(z_p^2+a^2\right)^{-3/2}\right]\hat{z} = \frac{kQz_p}{\left(z_p^2+a^2\right)^{3/2}}\hat{z}$$

(The differentiation must be with respect to the point (p) coordinates). Thus, the electric field along the symmetry axis is given by:

$$\vec{E} = \frac{kQz_p}{\left(z_p^2+a^2\right)^{3/2}}\hat{z}$$

This is pretty much the same electric field that we obtained earlier for the ring problem.

We now show an important connection between the electric field and the electric potential at a point in space. Since the electro**static** force is conservative, we have that the electrostatic field is related to a conservative potential by:

non-calculus: $\vec{E} = -\frac{\Delta V}{\Delta x}\hat{x} : \vec{E}$ is unifom in the x direction.

calculus: $\vec{E} = -\vec{\nabla}V$

where the differentiation is with respect to point (not charge) coordinates. Note however that later we will run into a type of electric field that is not conservative since it is generated by a changing magnetic field; this will not apply to that field.

Notice also that I have **bold faced** the static portion of electrostatic. This may not be true for time – dependent electric fields.

The gradient operator is the inverted delta and it is given by

(1): Cartesian

$$\vec{\nabla}T = \frac{\partial T}{\partial x}\hat{x} + \frac{\partial T}{\partial y}\hat{y} + \frac{\partial T}{\partial z}\hat{z}$$

(2) Spherical

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial T}{\partial \phi}\hat{\phi}$$

(3) Cylindrical

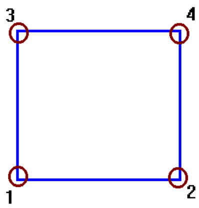
$$\vec{\nabla}T = \frac{\partial T}{\partial s}\hat{s} + \frac{1}{s}\frac{\partial T}{\partial \phi}\hat{\phi} + \frac{\partial T}{\partial z}\hat{z}$$

where T is an arbitrary scalar function involving position. Let's show this for the potential of a point charge developed above:

$$-\vec{\nabla} V = -\frac{\partial}{\partial r}\left(k\frac{q}{r}\right)\hat{r} = k\frac{q}{r^2}\hat{r} \Rightarrow \vec{E} = -k\frac{q}{r^2}\hat{r}$$

I will point out that there is a bit of a problem which will necessitate the introduction of a new type of mathematical function (called the "delta" function) if you look very closely at this expression, particularly when r=0. For now, that is not a problem.

Now let me show you one way to use the electrostatic potential of a point charge.



Let's determine how much work is required to assemble a charge distribution of equal charges in the form of a square with the following descriptions:

(1,q,0,0,0)
(2,q,1,0,0)
(3,q,0,1,0)
(4,q,1,1,0)

The total work required is the work to bring in each additional charge in the presence of charges already there:

$$W = W_{12} + W_{13} + W_{14} + W_{23} + W_{24} + W_{34}$$

This work required to assemble the charge distribution can be written as:

$$W = \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N}\frac{k q_i q_j}{\left|\vec{r}_j - \vec{r}_i\right|}$$

Now there is a relatively easy way to express this work: write it out as a square and then add up all the terms. Here is an example:

$$W = \frac{1}{2}\left[\begin{array}{ccccccc} [+]^{j\rightarrow}\!/_{i\downarrow} & 1 & 2 & 3 & 4 & \dots & N \\ 1 & 0 & W_{12} & W_{13} & W_{14} & \dots & W_{1N} \\ 2 & W_{21} & 0 & W_{23} & W_{24} & \dots & W_{2N} \\ 3 & W_{31} & W_{32} & 0 & W_{34} & \dots & W_{3N} \\ 4 & W_{41} & W_{42} & W_{43} & 0 & \dots & W_{4N} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ N & W_{N1} & W_{N2} & W_{N3} & W_{N4} & \dots & 0 \end{array}\right]$$

where my unusual sign [+] means that you are to add up everything inside the []. I've written it here just to be clear about what you are going to be doing with the []. Now an important property here is that if we use the entire thing in brackets, then we are counting the charges exactly twice too many times. We then can reduce the amount of our work required by recognizing that $W_{ij} = W_{ji}$, eliminating everything below the diagonal of zeros and multiplying by 2. The next step makes this look like the following:

$$W=\begin{bmatrix} [+]^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 & \ldots & N \\ 1 & 0 & W_{12} & W_{13} & W_{14} & \ldots & W_{1N} \\ 2 & 0 & 0 & W_{23} & W_{24} & \ldots & W_{2N} \\ 3 & 0 & 0 & 0 & W_{34} & \ldots & W_{3N} \\ 4 & 0 & 0 & 0 & 0 & \ldots & W_{4N} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ N & 0 & 0 & 0 & 0 & \ldots & 0 \end{bmatrix}$$

Now let's apply this to the specifics of the problem at hand. Each charge is the same and so the works involved for each charge ultimately require calculation of distances. The intermediate steps showing this looks like

$$: \quad W=\begin{bmatrix} [+]^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 \\ 1 & 0 & W_{12} & W_{13} & W_{14} \\ 2 & 0 & 0 & W_{23} & W_{24} \\ 3 & 0 & 0 & 0 & W_{34} \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$W=k\begin{bmatrix} [+]^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 \\ 1 & 0 & \frac{q_1 q_2}{|r_{12}|} & \frac{q_1 q_3}{|r_{13}|} & \frac{q_1 q_4}{|r_{14}|} \\ 2 & 0 & 0 & \frac{q_2 q_3}{|r_{23}|} & \frac{q_2 q_4}{|r_{24}|} \\ 3 & 0 & 0 & 0 & \frac{q_3 q_4}{|r_{34}|} \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We thus need to calculate the various distances involved. These are given by:

$$|\vec{r}_1-\vec{r}_2|=|\vec{r}_1-\vec{r}_3|=|\vec{r}_2-\vec{r}_4|=|\vec{r}_3-\vec{r}_4|=1 \quad \text{and} \quad |\vec{r}_1-\vec{r}_4|=|\vec{r}_2-\vec{r}_3|=\sqrt{2}$$

Also, in the particular problem at hand, each q is the same. Putting these inside the [] and factoring out the charge above then gives:

$$W = kq^2 \begin{bmatrix} [+] \; {}^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 \\ 1 & 0 & \frac{1}{1} & \frac{1}{1} & \frac{1}{\sqrt{2}} \\ 2 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{1} \\ 3 & 0 & 0 & 0 & \frac{1}{1} \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

You'll note I've eliminated everything but what is specifically involved in this example now. One small simplifying step gives:

$$W = kq^2 \begin{bmatrix} [+] \; {}^{j\rightarrow}/_{i\downarrow} & 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 & \frac{1}{\sqrt{2}} \\ 2 & 0 & 0 & \frac{1}{\sqrt{2}} & 1 \\ 3 & 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now the next step is to add up everything inside the []. This gives:

$$W = kq^2\left[1 + 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + 1 + 1\right] = kq^2\left[4 + \frac{2}{\sqrt{2}}\right] = kq^2\left[4 + \sqrt{2}\right] = 5.66\,kq^2$$

If you're just dealing with a few charges, it is probably easier to write out the individual work terms and adding them rather than using the double sum notation. I've written the double sum notation here to give you a method of attack that will always work when dealing with discrete point charges. I've also provided you with a framework for evaluating this general result in a nice ordered step-by-step procedure. The only variations might be that you will need to include charge values inside the [] if the charges are different (which was not the case in this example).

Once you have this potential energy, you can answer the questions of motion for any of the charges if you know the masses to which the charges are attached.

As you can see, electric potential is a very important quantity, physically. It is good, therefore, that you understand how to calculate it, especially in simple situations.

The electrostatic potential superimposes. Thus, the potential at a point in space is given by

$$V(\vec{r}_p) = \sum_{i=1}^{N} V_i(\vec{r}_p - \vec{r}_i)$$

Note that although the electrostatic potential superimposes, electrostatic energies do not always superimpose.

Let's look at one other way to write the work required to assemble a charge distribution.

$$W=\left[\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} k\frac{q_i q_j}{|\vec{r}_j-\vec{r}_i|}\right]=\frac{1}{2}\left[\sum_{i=1}^{N} q_i \sum_{\substack{j=1\\ j\neq i}} k\frac{q_j}{|\vec{r}_j-\vec{r}_i|}\right]=\frac{1}{w}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

Here is a quick example: suppose the potential at a point in space is 5V. What is the work required to bring a 0.5μC charge to this point? The answer is

$$W=\frac{1}{2}(0.5\,x\,10^{-6})\,x\,5=1.25x10^{-6}\,J$$

As you can see, knowing the potential at a particular point in space is useful. Of course, this is assuming that the action of bring the charge to that location does not, in fact, change the potential that you are working against. Later, we'll need a modification for the case where separating charge changes the potential (as in a capacitor). If you have a continuous charge distribution, then the calculus modification of this is:

$$W=\frac{1}{2}\int_{\substack{\text{all}\\ \text{charges}}} V(\vec{r})\,dq$$

Where $V(\vec{r})$ is the potential experienced at the particular charge dq which is being integrated over. What this says is that the work required to bring a charge to a point is the product of that charge and the electrostatic potential at the point in space where that charge is located.

Seeing as the electric potential is so important, we ought to see how to calculate it.

Suppose you have an infinitely long wire. The potential is given by:

$$V=\int_{a}^{r_p}\vec{E}\cdot d\vec{l}$$

You'll note something funny here .. I started at "a" .. for systems like long wires and infinite planes, you need to be careful not to start at infinity since that is exactly what will result. In these situations, it is better to just pick a point in space to start at which is not infinity. The question of why arises here. The answer is that actually the problem is not completely physical since your charge would actually be infinite that is on the wire.

What is E for the long wire? This is easily calculated by Gauss's Law:

$$\Phi_E=\sum_{\Delta A_i}\vec{E}_i\cdot\Delta\vec{A}_i=\sum_{\Delta A_i}\vec{E}_i\cdot\hat{n}\,\Delta A_i=\sum_{\Delta A_i}E_i\cdot\Delta A_i=E\sum_{\Delta A_i}A_i=E(2\pi sh)$$

$$\Phi=\oiint\vec{E}\cdot d\vec{A}=E(2\pi sh)$$

And

$$\frac{Q_{enc}}{\epsilon_0}=\frac{\lambda h}{\epsilon_0}$$

Thus, the electric field from the long wire is:

$$\vec{E}=\frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

The electric potential at s_p is then:

$$V=\sum_{r=a}^{r=s_p}\frac{\lambda}{2\pi\epsilon_0 s}\Delta s=\frac{\lambda}{2\pi\epsilon_0}\sum_{r=a}^{r=s_p}\frac{\Delta s}{s}=\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{S_p}{a}\right)$$

$$V=\int_{s=a}^{s_p}\frac{\lambda}{2\pi\epsilon_0 s}ds=\frac{\lambda}{2\pi\epsilon_0}\int_{s=a}^{s_p}\frac{ds}{s}=\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_p}{a}\right)$$

The problem here, of course, is that we can't reference the potential to infinity or zero so I chose to reference it to a point a. That ought to be ok … the potential is not completely determined except to within a constant. I suppose you could say that the potential is not directly measurable; it is the potential difference that is the measurable quantity.

What about the potential at points along the symmetry axis of the electric dipole? We had, for the dipole,

$$\vec{E}_p=2kq\left(\frac{q}{\left(a^2+y^2\right)^{3/2}}\right)\left(-\hat{x}\right) \quad .$$

So we can determine the potential as:

$$V=\oint_{path}\vec{E}\cdot d\vec{s}=\int_{\infty}^{r_p}-2kq\frac{a}{\left(a^2+y^2\right)^{3/2}}\hat{x}\cdot\hat{y}\,dy=0$$

(it's zero because the electric field points along the x-direction but the path taken is along the y-axis). The dot product is therefore zero. We ought to be able to get this from a direct superposition of the potentials due to two point charges also. Let's see if we can. For a point charge,

$$V=k\frac{q_i}{\left|\vec{r}_p-\vec{r}_i\right|}$$

Along the y-axis, we then have:

$$\vec{r}_p=0\hat{x}+y_p\hat{y};\vec{r}_1=-a\hat{x}+0\hat{y};\vec{r}_2=+a\hat{x}+0\hat{y}$$

So thus, the potential along the symmetry axis is given by:

$$V=kq\left(\frac{-1}{\sqrt{a^2+y_p^2}}+\frac{1}{\sqrt{a^2+y_p^2}}\right)=0$$

Hmmm .. of course we knew this …. If you're coming in along this axis, you are not doing work at all since the electric field points in the x direction but the displacement is in the y direction.

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(1) Find the work required to bring a charge q_p from infinity to a location $\vec{r}_p$ in the presence of an arbitrary charge distribution which produces at $\vec{r}_p$ an electrostatic potential V. Thus, use this to introduce the electrostatic potential.

Solution: From the definition of electrostatic potential we have:

$$V \equiv \frac{W}{q_p}$$

Provided you know what the electrostatic potential for the arbitrary charge distribution is, you are then able to find the work required to move a charge close to this distribution. The result is:

$$W = V(\vec{r}_p) q_p \quad .$$

Notice that this work can be either positive or negative, depending upon the potential and the charge.

(2) Find the electrostatic potential due to a conducting sphere of radius a (located at the origin) at a point $\vec{r}_p$ from the center of the sphere if the sphere has a total charge Q.

Solution: The sphere acts like a point charge for regions external to the sphere. From class notes, we know that the electrostatic potential of a point charge is given by:

$$V(\vec{r}_p) = k \frac{q_i}{|\vec{r}_p - \vec{r}_i|}$$

The sphere has a total charge Q and it is located at the origin. Thus, we have:

$$V(\vec{r}_p) = k \frac{Q}{|\vec{r}_p|}$$

At the surface of the sphere, the potential reaches its maximum value which is

$$V(a) = k \frac{Q}{a}$$

Since the sphere is conducting, all the charge will reside on its surface. The potential inside the sphere will be constant (**Not necessarily zero**) and equal to the value at the surface.

Calculus: Determine the electric field from the potential of the sphere.

$$\vec{E} = -\vec{\nabla}_p V(\vec{r}_p) \hat{r}_p = \frac{-\partial}{\partial r_p} \left(k \frac{Q}{\sqrt{r_p^2}} \right) \hat{r}_p = -kQ \left(\frac{\partial}{\partial r_p} (r_p^2)^{-1/2} \right) \hat{r}_p = kQ \frac{1}{2} 2 r_p (r_p^2)^{-3/2} \hat{r}_p = \frac{kQ}{r_p^2} \hat{r}_p$$

What do you imagine we'd do if the sphere were not located at the origin?
The potential for a sphere not at the origin is given by:

$$V(\vec{r}_p) = k \frac{Q}{|\vec{r}_{ip}|}$$

The electric field is still given by:

$$\vec{E}_p = -\vec{\nabla}_p V(\vec{r}_{ip})$$

The problem here is to take the gradient. Let me do this in Cartesian coordinates.

$$-\vec{\nabla}_p \frac{1}{|\vec{r}_p - \vec{r}_i|} = -\left[\frac{\partial}{\partial x_p}\hat{x} + \frac{\partial}{\partial} y_p \hat{y} + \frac{\partial}{\partial z_p}\hat{z}\right]\left[(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2\right]^{-1/2}$$

$$= \frac{1}{2}\left[(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2\right]^{-3/2}\left[2(x_p - x_i)\hat{x} + 2(y_p - y_i)\hat{y} + 2(z_p - z_i)\hat{z}\right]$$

$$= \frac{1}{|\vec{r}_{ip}|^2}\frac{\vec{r}_{ip}}{|\vec{r}_{ip}|} = \frac{\hat{r}_{ip}}{|\vec{r}_{ip}|^2} \Rightarrow \vec{E}_p = \frac{kQ}{|\vec{r}_{ip}|^2}\hat{r}_{ip}$$

It takes a bit of thought to realize that $\left[(x_p - x_i)\hat{x} + (y_p - y_i)\hat{y} + (z_p - z_i)\hat{z}\right] = \vec{r}_{ip}$ but the key to understanding this is to understand that the Cartesian unit vectors are the same for the “i” coordinates as for the “p” coordinates.

In passing, I’ll just note that the same could have been accomplished by defining a more general gradient operator as:

$$\vec{\nabla}_{ip} = \frac{\partial}{\partial(x_p - x_i)}\hat{x} + \frac{\partial}{\partial(y_p - y_i)}\hat{y} + \frac{\partial}{\partial(z_p - z_i)}\hat{z}$$

(3) Determine the work required to assemble 3 charges given by:

(#,q,x,y,z)=(1,3μC,1,1,1),(2,6μC,2,2,2),(3,-9μC,3,3,3).

Solution: from class notes, we have:

$$W = \left[\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1 \\ j\neq i}}^{N} k\frac{q_i q_j}{|\vec{r}_j - \vec{r}_i|}\right] = k\begin{bmatrix} [+] \; {}^{j\rightarrow}\!/_{i\downarrow} & 1 & 2 & 3 \\ 1 & 0 & \frac{q_1 q_2}{|r_{12}|} & \frac{q_1 q_3}{|r_{13}|} \\ 2 & 0 & 0 & \frac{q_2 q_3}{|r_{23}|} \end{bmatrix} = k\mu\mu\begin{bmatrix} \frac{18}{\sqrt{3}} & \frac{-27}{\sqrt{12}} \\ 0 & \frac{-54}{\sqrt{3}} \end{bmatrix}$$

$$= k\mu\mu\left(\frac{18}{\sqrt{3}} - \frac{27}{\sqrt{12}} - \frac{54}{\sqrt{3}}\right) = 8.99\text{x}10^{-3}(10.392 - 7.794 - 31.717) = -0.262\,\text{J}$$

Notice that the last row can actually be omitted because it is filled with zero’s. You will also note another difference: we need to leave the charges inside the [] this time since they are not all of the same magnitude. Notice, however, that since each charge is micro, I am able to factor out that exponent.

(4) The potential difference is defined as the difference in electrostatic potential between two points in space. It can also be defined as the negative work per unit charge to move a charge between two points in space.
Determine the potential difference between two plates of a ideal parallel plate capacitor with plates of area A and separation d which has a charge separation Q on one plate and –Q on the other plate. **Note: this is a very important problem. Make sure you understand it.**
Solution:
From Gauss's law, we have

$$\vec{E}=\frac{\sigma}{\epsilon_0}\hat{x}$$

between the plates of the capacitor if the positive plate is located at the origin.
Since the electric field is uniform, we can easily determine the potential difference (which is often written V):

$$\Delta V=-\vec{E}\cdot\Delta\vec{x}=-\frac{\sigma d}{\epsilon_0}$$

Since the plates have an area A, and a total charge separation Q , we thus have

$$\Delta V=-\frac{Qd}{A\epsilon_0}$$

The capacitance of the capacitor is defined as the (magnitude) of the ratio of the charge separation to the potential. Thus:

$$C\equiv\left|\frac{Q}{\Delta V}\right|=\frac{Q}{\left(\frac{Qd}{A\epsilon_0}\right)}=\epsilon_0\frac{A}{d}$$

for the parallel plate capacitor. In passing, I'd like to mention that although we speak of "charging up a capacitor", in fact what is happening is that an electrically neutral capacitor has the charge separated from one plate to another. The capacitor is overall electrically neutral before charging and also after charging. You have, however, moved a total charge Q from one plate and placed it on the other plate.

(5) A parallel plate capacitor of capacitance C=6μf is charged to a potential difference of 200 Volts. How much energy is stored in the capacitor? If the plate separation is 0.0001m, what is the energy density of the capacitor? Then, what is the magnitude of the electric field inside the capacitor? **Note: this is a very important problem. Make sure you understand it. The concept of energy density is also extremely important to understand.**

Solution:

The potential that a given charge moves against is given by:

$$\Delta V=\frac{Q}{C}$$

where Q represents the charge that has already been moved from one plate to the other. The amount of work that is required to move a single charge q across this potential difference is given by:

$$W=q\Delta V=q\frac{Q}{C}$$

But now you notice that the potential difference across the capacitor has increased. In fact, the potential difference is now:

$$\Delta V=\frac{Q+q}{C}$$

The work required to move the next q across the capacitor is then given by:

$$W=q\frac{Q+q}{C}$$

As you can see, the work for each successive charge increases. Let me try to formulate a sequence for this (Each q is the charge on the electron or proton here ... it is really small). I am starting with a completely uncharged capacitor and moving equal sized elementary charges across one at a time, and adding up the work done for each.

$$W=0+q\frac{q}{C}+q\frac{2q}{C}+q\frac{3q}{C}+q\frac{4q}{C}+...+q\frac{Nq}{C}=\frac{q^2}{C}[0+1+2+3+...+N]=\frac{q^2}{C}\sum_{i=1}^{i=N} i$$

This is an arithmetic series. You can look up arithmetic series at Mathworld and you will find:

$$\sum_{i=1}^{i=N} i=\frac{1}{2}n(n+1)$$

Now if n is large (it is, typically) then the sequence can be approximated as:

$$\sum_{i=1}^{i=N} i\approx\frac{1}{2}n^2$$

Here are some quick examples to show the validity of this approximation:

$$1+2+3+4+5=5+5+5=15=\frac{1}{2}(5)(6)=\frac{30}{2}=15:\frac{1}{2}(5^2)=12.5\Rightarrow 16.7\,\%\text{error}$$

$$\sum_{i=1}^{100} i=\frac{1}{2}(100)(101)=5050:\frac{1}{2}(100^2)=5000\Rightarrow 1\,\%\text{error}$$

Normally n might be on the order of 10^{10} (or more). You can see then that the approximation is pretty much valid as n gets larger (it is already good at 100). However, with the new nanotechnology advancements, you may need to get away from the approximation and add up discrete charges like this. Of course, there are still approximations involved for the potential between the plates of the capacitor. A more detailed analysis would show that the charges on the plates also spread out as much as they can so that the potential for the first few charges in fact is different than what I have represented. I'll not worry about that here. So we have the work required to be given by:

$$W = \frac{q^2}{C}\left[\frac{1}{2}n^2\right] = \frac{1}{2}\frac{(nq)^2}{C} = \frac{Q^2}{2C}$$

Now let me show you a second approach that will give the same important result:

$$\Delta V = \frac{Q}{C}$$

The real difficulty here lies with the fact that the charges are moving against a changing potential … every charge moved across the plates makes it harder for the next charge to be moved across the plates.

Non-calculus: We can (in a non-calculus way) answer this question then by saying that Q is moved against the average V (since the potential is linear in q). Thus,

$$W = Q(\Delta V)_{average} = \frac{Q^2}{2C}$$

Calculus: The total work is given by:

$$W = \int_{q=0}^{Q} \frac{q}{C}\,dq = \frac{Q^2}{2C}$$

Thus, the total energy stored in the capacitor will be given by: (Hint: Very important!)

$$U = \frac{Q^2}{2C} = \frac{1}{2}C(\Delta V)^2$$

Now, how much energy is stored in our capacitor?

$$U = \frac{1}{2}6\times10^{-6}(200)^2 = 0.12\,J$$

Now for the rest of the formulation … Since for a parallel plate capacitor, we have

$$C = \epsilon_0 \frac{A}{d}$$

we can write the stored energy as:

$$U = \frac{1}{2}\epsilon_0 \frac{A}{d}(Ed)^2 = \frac{1}{2}\epsilon_0 (Ad)E^2$$

The term Ad is the internal volume of the capacitor. Thus, the ***energy density*** of the capacitor is given by

$$u \equiv \frac{U}{volume} = \frac{1}{2}\epsilon_0 E^2$$

For our present capacitor, we can find the plate area from:

$$A=\frac{Cd}{\epsilon_0}=\frac{6x10^{-6}x0.0001}{8.854x10^{-12}}=67.8\,m^2$$

The energy density is $u=\frac{0.12\,J}{67.8x0.0001\,m^3}=17.7\,\frac{J}{m^3}$

The electric field inside is

$$E=\frac{V}{d}=\frac{200V}{0.0001\,m}=2x10^6\,\frac{V}{m}$$

Clearly, there has got to be more to the story than I have told you. The rest involves inserting materials into the capacitor with a high dielectric constant.