

Integral Calculus 1.3

1.3.1 Line, Surface and Volume Integrals

(a) **Line Integrals:** A line integral (=path integral, =contour integral) appears like this, although your author does use slightly different notation.

$$\oint_a^b \vec{v} \cdot d\vec{L}$$

Your author uses the circle notation to indicate a closed path.

I should but I probably won't. His notation is this

$$\text{for a path that is not closed: } \int_p \vec{v} \cdot d\vec{L} .$$

The integral consists of two important parts: a path vector and a function vector.

It is important to understand just how to produce the path vector for a desired path.

Consider Example 1.6:

$$\vec{v} = y^2 \hat{x} + 2x(y+1) \hat{y}.$$

It is desired to integrate this along the path from (1,1,0) to (2,2,0) .

path 1: straight lines: (1,1,0) to (2,1,0) then from (2,1,0) to (2,2,0).

Along the first portion: always start: $d\vec{L} = dx \hat{x} + dy \hat{y} + dz \hat{z}$

the curve for the first portion is described by:

$$y = \text{constant} = 1 : x = x \Rightarrow dy = 0 : dx = dx \Rightarrow d\vec{L} = dx \hat{x} + 0 \hat{y} + 0 \hat{z}$$

Next take the dot product:

$$\vec{v} \cdot d\vec{L} = y^2 dx$$

Now place it in the integral and integrate.

$$\text{Path 1: } \int_{(1,1,0)}^{(2,1,0)} y^2 dx = xy^2 \Big|_{(1,1,0)}^{(2,1,0)} = 2 - 1 = 1$$

Along the first portion: always start: $d\vec{L} = dx \hat{x} + dy \hat{y} + dz \hat{z}$

the curve for the first portion is described by:

$$x = \text{constant} = 2 : y = y \Rightarrow dx = 0 : dy = dy \Rightarrow d\vec{L} = 0 \hat{x} + dy \hat{y}$$

Next take the dot product:

$$\vec{v} \cdot d\vec{L} = 2x(y+1)$$

Now place it in the integral and integrate.

$$\text{Path 2: } \int_{(2,1,0)}^{(2,2,0)} 2x(y+1) dy = 2x \left(\frac{y^2}{2} + y \right) \Big|_{(2,1,0)}^{(2,2,0)} = 4(2+2) - 4 \left(\frac{1}{2} + 1 \right) = 16 - 4 \left(\frac{3}{2} \right) = 12 - 6 = 10$$

So the total result is obtained by adding the two parts which is 11.

Now consider the path which is a straight line from (1,1,0) to (2,2,0).

Along the first portion: always start: $d\vec{L}=dx\hat{x}+dy\hat{y}+dz\hat{z}$
 the curve for the first portion is described by: ... ok, we need to find the equation of the line now.

A straight line has the form:

$$y=mx+b$$

$$\text{The slope is given by: } m=\frac{\Delta y}{\Delta x}=\frac{2-1}{2-1}=1$$

Now to get the intercept:

$$y=x+b: @x=1, y=1 \Rightarrow 1=1+b \Rightarrow b=0$$

So the line has the equation $y=x \Rightarrow dy=dx$

So we can write:

$$d\vec{L}=dx\hat{x}+dy\hat{y}+0\hat{z}$$

and so:

$$\vec{v} \cdot d\vec{L}=y^2 dx+2x(y+1)dx=x^2 dx+2x(x+1)dx=(x^2+2x^2+2x)dx$$

$$\vec{v} \cdot d\vec{L}=(3x^2+2x)dx$$

$$\oint_a^b \vec{v} \cdot d\vec{L}=\int_{x=1}^{x=2} (3x^2+2x)dx=(x^3+x^2)|_1^2=(8+4-2)=10$$

Finally if you go along the first path, then the second path then come back the second way, the result for the closed path is 11-10=1

This function then is also not conservative.

Let's try one more path: go from (1,1,0) to (2,2,0) along $y=x^2$.

Along the first portion: always start: $d\vec{L}=dx\hat{x}+dy\hat{y}+dz\hat{z}$

To formulate the path element: (it's not always this easy)

$$dy=2xdx \Rightarrow d\vec{L}=dx\hat{x}+2xdx\hat{y}$$

$$\vec{v} \cdot d\vec{L}=(y^2\hat{x}+2x(y+1)\hat{y}) \cdot (dx\hat{x}+2xdx\hat{y})=y^2 dx+2x(y+1)2xdx$$

$$\Rightarrow \vec{v} \cdot d\vec{L}=x^4 dx+2x(x^2+1)2xdx=(x^4+4x^4+4x^2)dx=(5x^4+4x^2)$$

so we then have:

$$\oint_{(1,1,0)}^{(2,2,0)} \vec{v} \cdot d\vec{L}=(x^5+\frac{4}{3}x^3) \Big|_{(1,1,0)}^{(2,2,0)}=(32+\frac{4}{3}8)-(1+\frac{4}{3})=31+7\frac{4}{3}=40.3333$$

Surface Integrals

Your author uses this notation for a closed surface integral: $\oint_a^b \vec{v} \cdot d\vec{a}$.

My notation looks more like this for a closed or open surface integral:
 $\oiint \vec{v} \cdot d\vec{a}$

Your author use this notation for an open surface integral: $\int_S \vec{v} \cdot d\vec{L}$
 a represents an elementary patch of area.

Let's look at example 1.7: but I'll just do it over one side.

$$\vec{v} = 2xz\hat{x} + (x+2)\hat{y} + y(z^3 - 3)\hat{z}$$

Let's look at the side described by $x=2$. Then on this side:

$$d\vec{a} = dydz\hat{x}$$

(it points in the $+x$ direction for a right handed coordinate system).

On this side then, we have:

$$\oiint \vec{v} \cdot d\vec{a} = \oiint 2xz dydz = 4 \int_{z=0}^{z=2} \int_{y=0}^{y=2} z dydz = 4 \left(\frac{z^2}{2} y \right)_{(y=0, z=0)}^{(y=2, z=2)} = 16 - 0 = 16$$

To complete this example, you would integrate over the other cube faces and then add the results to produce the final answer.

Volume Integrals

Your author uses this notation for volume integrals: $\int_V T d\tau$ where T can be a scalar function or a vector function and $d\tau$ is an elementary section of volume. Note right at the beginning an important thing though: when dealing with vector integrals, make sure that if the result is to be a vector, only use Cartesian unit vectors since these are the only unit vectors that are constant and can be taken outside of the integrals. My notation for the volume integral is this:

$$\iiint T d^3R \quad \text{or} \quad \iiint T d\tau .$$

Let's look at example 1.8 in the text:

$T=xyz^3$ and you are integrating over the prism in figure 1.24.

We need to calculate the limits on x to integrate over the face of the prism. The equation of the line in the x - y plane is $y=-x+1$. This means that the limits on x at a particular y are given by $x=1-y$. I thought I'd do it slightly different than the example though so I'll use the limits on y rather than x . Our required integral is:

$$\begin{aligned} \iiint (xyz^2) d\tau &= \int_{z=0}^z \int_{x=0}^x \int_{y=0}^{y=1-x} (xyz^2) dz dx dy = \int_{z=0}^z z^2 dz \int_{x=0}^x x \int_{y=0}^{y=1-x} y dx dy \\ &\Rightarrow \iiint (xyz^2) d\tau = \left(\frac{z^3}{3}\right)_{z=0}^z \int_{x=0}^x x dx \left(\frac{y^2}{2}\right)_{y=0}^{y=1-x} = (9) \frac{1}{2} \int_{x=0}^x x dx (1-2x+x^2) \\ &\Rightarrow \iiint (xyz^2) d\tau = \frac{9}{2} \left(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right)_{x=0}^x = \frac{9}{2} \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) = \frac{9}{2} \left(\frac{6}{12} - \frac{8}{12} + \frac{3}{12}\right) = \frac{9}{2} \left(\frac{1}{12}\right) = \frac{3}{8}. \end{aligned}$$

which is the same as your author obtained.