Differential Calculus

1.2.1 Ordinary derivatives

the derivative of a function tells how rapidly the function changes in an infinitesimal change and is interpreted as the slope of the graph of f vs. x.

$$df=\left(\frac{df}{dx}\right)dx$$

1.2.2 Gradient

In more than one dimension, for a scalar function f, we have:

$$df=\left(\frac{\partial f}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}\right)dz$$

which is the answer to "how fast does f vary?"

we define the gradient of f as:

$$\vec{\nabla} f\equiv\frac{\partial f}{\partial x}\hat{x}+\frac{\partial f}{\partial y}\hat{y}+\frac{\partial f}{\partial z}\hat{z}$$

Then the derivative of f is given by:

$$df=\vec{\nabla} f\cdot d\vec{l}$$

The gradient of f points in the direction of maximum increase of f and its magnitude gives the slope along this direction.

Example 1.3

The magnitude of the position vector is given by $r=\sqrt{x^2+y^2+z^2}$

$$\vec{\nabla} f=\frac{x}{\sqrt{x^2+y^2+z^2}}\hat{x}+\frac{y}{\sqrt{x^2+y^2+z^2}}\hat{y}+\frac{z}{\sqrt{x^2+y^2+z^2}}\hat{z}$$

$$\vec{\nabla} f=\frac{\vec{r}}{|\vec{r}|}\equiv\hat{r}$$

$$\left|\vec{\nabla} f\right|=\sqrt{\vec{\nabla} f\cdot\vec{\nabla} f}=1$$

So the gradient of the position vector gives a unit vector in that direction.

Problem 1.13

Let $\vec{r}_{ip}$ be the separation vector pointing from a source point $\vec{r}_i=x_i\hat{x}+y_i\hat{y}+z_i\hat{z}$ to the field point $\vec{r}_p=x_p\hat{x}+y_p\hat{y}+z_p\hat{z}$. Show that (a) $\vec{\nabla}(r_{ip}^2)=2\,\vec{r}_{ip}$, $\vec{\nabla}\left(\frac{1}{r_{ip}}\right)=\frac{-\hat{r}_{ip}}{r_{ip}^2}$,

and determine the general formula for $\vec{\nabla}(r_{ip}^n)$.

First obtain the required vector.

Note that correctly and clearly stated, I should say:

$\vec{\nabla}_p(r_{ip}^2)=2\,\vec{r}_{ip}$, $\vec{\nabla}_p\left(\frac{1}{r_{ip}}\right)=\frac{-\hat{r}_{ip}}{r_{ip}^2}$, and $\vec{\nabla}_p(r_{ip}^n)$

which means take the operations with respect to the space coordinates, and not the charge coordinates.

It is worthwhile for you to compare this:

$\vec{\nabla}_p|\vec{r}_{ip}|$ $\vec{\nabla}_i|\vec{r}_{ip}|$ where $|\vec{r}_{ip}|=\sqrt{(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2}$.

$$\vec{r}_{ip}=\vec{r}_p-\vec{r}_i=(x_p-x_i)\hat{x}+(y_p-y_i)\hat{y}+(z_p-z_i)\hat{x}$$

$$r_{ip}^2=\vec{r}_{ip}\cdot\vec{r}_{ip}=(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2$$

$$\vec{\nabla}_p(r_{ip}^2)=\frac{\partial r_{ip}^2}{\partial x_p}\hat{x}+\frac{\partial r_{ip}^2}{\partial y_p}\hat{y}+\frac{\partial r_{ip}^2}{\partial z_p}\hat{z}$$

$$\vec{\nabla}_p(r_{ip}^2)=2(x_p-x_i)\hat{x}+2(y_p-y_i)\hat{y}+2(z_p-z_i)\hat{z}=2\,\vec{r}_{ip}$$

$$\begin{aligned}
&\frac{\partial[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(-1/2)}}{\partial x_p}\hat{x}\\
\vec{\nabla}_p\left(\frac{1}{r_{ip}}\right)=&+\frac{\partial[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(-1/2)}}{\partial y_p}\hat{y}\\
&+\frac{\partial[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(-1/2)}}{\partial z_p}\hat{z}
\end{aligned}$$

$$\begin{aligned}
&(-1/2)2(x_p-x_i)[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(-3/2)}\hat{x}\\
\vec{\nabla}_p\left(\frac{1}{r_{ip}}\right)=&+(-1/2)2(y_p-y_i)[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(-3/2)}\hat{y}\\
&+(-1/2)2(z_p-z_i)[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(-3/2)}\hat{z}
\end{aligned}$$

$$\vec{\nabla}_p\left(\frac{1}{r_{ip}}\right)=-\frac{\vec{r}_{ip}}{[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(3/2)}}$$

$$\vec{\nabla}_p\left(\frac{1}{r_{ip}}\right)=-\frac{\vec{r}_{ip}}{[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(3/2)}}$$

$$\vec{\nabla}_p\left(\frac{1}{r_{ip}}\right) = -\frac{\vec{r}_{ip}}{\left|\vec{R}_{ip}\right|\vec{r}_{ip}^{\,2}} = \frac{-\hat{r}_{ip}}{\vec{r}_{ip}^{\,2}}$$

$$\vec{\nabla}_p\left(r_{ip}^{n}\right) = n\,r_{ip}^{n-1}\,\hat{r}_{ip}$$

Do this by taking the derivative with respect to each component.

1.2.4 Divergence

Let $\vec{T}=\vec{T}(\vec{r}_p)$.

$$\vec{\nabla}\cdot\vec{T}=\left(\hat{x}\frac{\partial}{\partial x_p}+\hat{y}\frac{\partial}{\partial y_p}+\hat{z}\frac{\partial}{\partial z_p}\right)\cdot\left(T_x\hat{x}+T_y\hat{y}+T_z\hat{z}\right)=\frac{\partial T_x}{\partial x_p}+\frac{\partial T_y}{\partial y_p}+\frac{\partial T_z}{\partial z_p}\equiv\vec{\nabla}_p\cdot\vec{T}$$

Which should clearly say that the differential operators do not act on the unit vectors, and the p subscript on the divergence operator emphasizes the point that these are all with regard to space points p. This also emphasizes that the divergence of a vector is a scalar just as is the dot product between two vectors.

Geometrically the divergence measures how a vector field spreads out from the point of interest. All the vector field really needs to do is be directed either away or towards a point, and not be uniform as you leave the point, essentially.

One extremely important application of the divergence is in **problem 1.16** which I want to do for you now.

Calculate the divergence of $\vec{v}=\frac{\hat{r}}{r^2}$. I'll assume that r depends upon p but it is not the separation vector unless the source is at the origin. I believe, though, what I really want to do is a modification of this problem, namely calculate the divergence of $\vec{v}=\frac{\hat{r}_{ip}}{r_{ip}^2}$ using the divergence operator $\vec{\nabla}_p$ that I defined above.

To do this problem, we need to calculate the various vectors first.

$$\vec{r}_i=x_i\hat{x}+y_i\hat{y}+z_i\hat{z}:\vec{r}_p=x_p\hat{x}+y_p\hat{y}+z_p\hat{z}:\vec{r}_{ip}=\vec{r}_p-\vec{r}_i=(x_p-x_i)\hat{x}+(y_p-y_i)\hat{y}+(z_p-z_i)\hat{z}$$

Then $\hat{r}_{ip}=\frac{(x_p-x_i)\hat{x}+(y_p-y_i)\hat{y}+(z_p-z_i)\hat{z}}{\sqrt{(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2}}$

So the vector is written as:

$$\vec{v}=\frac{(x_p-x_i)\hat{x}+(y_p-y_i)\hat{y}+(z_p-z_i)\hat{z}}{[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(3/2)}}$$

Each term of the divergence looks like:

$$\frac{\partial v_x}{\partial x_p}\hat{x}$$

so I'll do it for one and write down the other terms

$$\frac{\partial v_x}{\partial x_p}=\frac{1}{[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(3/2)}}-\frac{3(x_p-x_i)^2}{[(x_p-x_i)^2+(y_p-y_i)^2+(z_p-z_i)^2]^{(5/2)}}$$

so, considering the other two terms and adding their results, we have:

$$\vec{\nabla}_p \cdot \frac{\hat{r}_{ip}}{r_{ip}^2} = \frac{3}{\left|\vec{r}_{ip}\right|^3} - \frac{3}{\left|\vec{r}_{ip}\right|^3} = 0$$

This is, however, not the whole story. We had an earlier result that

$$\vec{\nabla}\left(\frac{1}{r_{ip}}\right) = -\frac{\vec{r}_{ip}}{\left|\vec{R}_{ip}\right| \vec{r}_{ip}^{\,2}} = \frac{-\hat{r}_{ip}}{\vec{r}_{ip}^{\,2}}$$

so this can be written as:

$$\vec{v} = -\vec{\nabla}\left(\frac{1}{r_{ip}}\right)$$

With the above results, this seems to imply:

$$\vec{\nabla}_p \cdot \vec{v} = -\vec{\nabla}_p \cdot \vec{\nabla}\left(\frac{1}{r_{ip}}\right) = 0$$

which is still not the whole story since in fact the divergence is infinite when

$$\vec{r}_p - \vec{r}_i = \vec{0} \quad .$$

This fact is central to the development of electrostatics.

Section 1.2.5 The Curl

The differential operator curl is defined by:

$$\vec{\nabla}_p \times \vec{v} \equiv \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x_p} & \frac{\partial}{\partial y_p} & \frac{\partial}{\partial z_p} \\ v_x & v_y & v_z \end{bmatrix} = \hat{x}\left(\frac{\partial v_z}{\partial y_p} - \frac{\partial v_y}{\partial z_p}\right) - \hat{y}\left(\frac{\partial v_z}{\partial x_p} - \frac{\partial v_x}{\partial z_p}\right) + \hat{z}\left(\frac{\partial v_y}{\partial x_p} - \frac{\partial v_x}{\partial y_p}\right)$$

The curl is central to magnetostatics where as the divergence is central to electrostatics.

Section 1.3 Integral Calculus