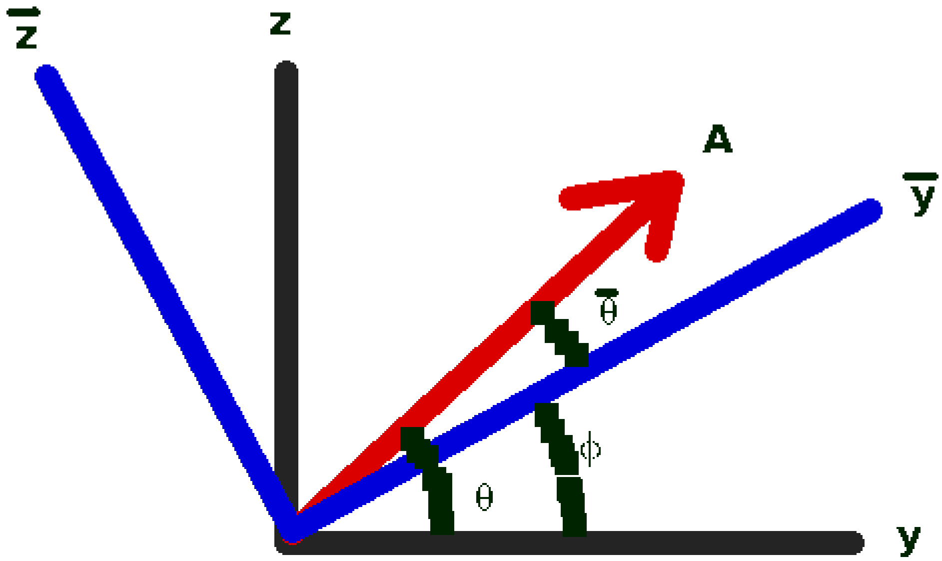
Section 1.1.5 How Vectors Transform 2016 rev 01

Vectors transform according to a very specific rule



We want to represent the vector A in both coordinate frames.

In the y-z plane: In the y - z plane: $A_y = |\vec{A}| \cos\theta : A_z = |\vec{A}| \sin\theta$

In the $\bar{y} - \bar{z}$ plane: $\bar{A}_{\bar{y}} = |\vec{A}| \cos\bar{\theta} : \bar{A}_{\bar{z}} = |\vec{A}| \sin\bar{\theta}$

(I believe the redundent notation is an improvement over the notation on P. 10)

But note: $\bar{\theta} = \theta - \phi$

And, since:

$$\cos\bar{\theta} = \cos(\theta - \phi) = \cos\theta\cos\phi + \sin\theta\sin\phi$$
$$\sin\bar{\theta} = \sin(\theta - \phi) = \sin\theta\cos\phi - \cos\theta\sin\phi$$

we have the transformed components:

$$\bar{A}_{\bar{y}} = |\vec{A}|(\cos\theta\cos\phi + \sin\theta\sin\phi) = A_y\cos\phi + A_z\sin\phi$$
$$\bar{A}_{\bar{z}} = |\vec{A}|(\sin\theta\cos\phi - \cos\theta\sin\phi) = A_z\cos\phi - A_y\sin\phi$$

This can be written in matrix form:

$$\begin{bmatrix} \bar{A}_{\bar{y}} \\ \bar{A}_{\bar{z}} \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} A_y \\ A_z \end{bmatrix}$$

According to your author, the form for rotation about an arbitrary axis in three dimensions is given by:

$$\begin{bmatrix} \bar{A}_{\bar{x}} \\ \bar{A}_{\bar{y}} \\ \bar{A}_{\bar{z}} \end{bmatrix} = \begin{bmatrix} R_{xx} & R_{xy} & R_{xz} \\ R_{yx} & R_{yy} & R_{yz} \\ R_{zx} & R_{zy} & R_{zz} \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Which is compactly expressed as

$$\bar{A}_{\bar{i}}=\sum_{j=1}^{3} R_{ij} A_j : 1\equiv x, 2\equiv y, 3\equiv z.$$

So formally a vector is a set of 3 coordinates that transforms in the same way as a displacement when coordinates are changed.

Problem 1.10 Page 12 (pseudovectors and pseudoscalars)

(a) Find vector transformation under Translation of coordinates :

$$\bar{x}=x, \bar{y}=y-a, \bar{z}=z$$

It is easiest to look at a real simple example first. $\vec{A}=0\hat{x}+5\hat{y}+0\hat{z}$ corresponds to only a point (5) along the y axis. The particular transformation for y given above moves y along the +y axis by +a units. Suppose that a was exactly 5. then in the transformed system, we would have:

$$\bar{\vec{A}}=0\hat{\bar{x}}+0\hat{\bar{y}}+0\hat{\bar{z}}$$

The unit vectors are, of course, the same in the two coordinate systems.

So the rule for transforming by translation is this:

$$\bar{\vec{A}}=(A_x-a_x)\hat{x}+(A_y-a_y)\hat{y}+(A_z-a_z)\hat{z}$$

for a generalized translation along vector given by:

$$\vec{a}=a_x\hat{x}+a_y\hat{y}+a_z\hat{z}$$

In the present problem, for a single translation along y, the result is:

$$\bar{\vec{A}}=x\hat{x}+(y-a)\hat{y}+z\hat{z}$$

The results (answer) is in fact that the vector is not invariant under translations. However, consider that A represents velocity. Then no matter where the coordinates are translated, each observer would see the same velocity. However, if A represents position, then in fact the components of A would change. Position seems to be a unique type of vector in this regard; however position does not often appear in physics, instead we talk about change in position or also directed distance.

A simple yes or no here does not suffice actually.

"Be very careful about distinguishing between points and vectors. Otherwise you are likely to get bitten by translations doing the Wrong Thing."

$$\bar{\vec{A}}=-\vec{a}+\vec{A} : \frac{d\bar{\vec{A}}}{dt}=-\frac{d\vec{a}}{dt}+\frac{d\vec{A}}{dt}=0+\frac{d\vec{A}}{dt}=\frac{d\vec{A}}{dt}$$

Quantities derived from position vectors are invariant under a translation. However a position vector itself is not invariant under a translation which some interpret as meaning that position is, in fact, a slightly different type of vector than other vectors. However I would contend that the velocity vector would also not be invariant under a translation at a constant velocity.

To really understand this completely, we need to look at the definition of a vector: it has magnitude and direction (and is not invariant under inversion). So while the coordinates of a point change under translation, the distance between two points does not change under translation. So when we speak about a vector in n-dimensional space, and although the vector is normally written as if it were only those n-coordinate points, in fact a vector of those n-points is referenced to the origin and the components of the vector refer to the distance between the origin and that coordinate. Otherwise, the vector does not translate properly.

Vectors are also expressed in other ways and perhaps here is the correct compact way to express them.

A vector can be defined as a set of n numbers $A_{1,} A_{2,} \ldots A_n$ that transform according to the rule $A_i^{'} = a_{ij} A_j$ where $a_{ij} = \frac{\partial x_i^{'}}{\partial x_j}$ (Einstein's summation notation) ... http://www.mathematicsmagazine.com/4-2005/Gr12_4_2005.htm (fixing 1 small error)

are constants (corresponding to the direction cosines), with partial derivatives taken with respect to the original and transformed coordinate axes, and i, j=1, ...,*n* (Arfken 1985, p. 10).

. This means that for this definition of a vector, there can be no doubt that the vector must be invariant under a translation. Arfken 1985, p. 10

This makes a vector a tensor of tensor rank one. A vector with *n* components in called an *n*-vector, and a scalar may therefore be thought of as a 1-vector (or a 0-tensor rank tensor). Vectors are invariant under translation, and they reverse sign upon inversion. Objects that resemble vectors but do not reverse sign upon inversion are known as pseudovectors. To distinguish vectors from pseudovectors, the former are sometimes called polar vectors.

(b) Find vector transformation under inversion of coordinates:

$$\bar{x} = -x, \bar{y} = -y, \bar{z} = -z$$

I think this is easiest to do step-by-step here after looking at a simple example or two. Consider the point (x,y,z)=(1,2,3). Then flipping of coordinates would make this look like (-1, -2,-3) since the requirement is:

$$\bar{x} = -x \Rightarrow -\bar{x} = 1 \Rightarrow \bar{x} = -1$$
$$\bar{y} = -y \Rightarrow -\bar{y} = 2 \Rightarrow \bar{y} = -2$$
$$\bar{z} = -z \Rightarrow -\bar{z} = 3 \Rightarrow \bar{z} = -3$$

Ok, so the way to proceed is this: the inversion transformation matrix and vector operation would be:

$$\begin{bmatrix} \bar{A}_x \\ \bar{A}_y \\ \bar{A}_z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

(c) Now consider the cross product of two vectors, A and B

$$\vec{C} \equiv \vec{A} X \vec{B} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{bmatrix} = (A_y B_z - B_y A_z)\hat{x} + (B_x A_z - A_x B_z)\hat{y} + (A_x B_y - B_x A_y)\hat{z}$$

The transformation requirement is in the inverted coordinate system, each component is replaced by it's negative. This under inversion:

$$\bar{\vec{C}} = \vec{C}$$

In order to investigate the transformation of the result of the cross product of two pseudovectors, we need to look at triple products and the extension; in particular the vector triple product, $\vec{A}\ x\ (\vec{B}\ x\ \vec{C})$. This is evaluated with the "BAC-CAB" rule most easily which is (and is provable, though a bit tedious) ...

$$\vec{A}\ x\ (\vec{B}\ x\ \vec{C})=\vec{B}(\vec{A}\cdot\vec{C})-\vec{C}(\vec{A}\cdot\vec{B})$$

It is easy to see from here that the results of the vector triple product of three vectors is to produce a vector because it is not invariant under an inversion transformation. We also have the possibility to use the "BAC-CAB" rule to evaluate higher order products to show:

$$(\vec{A}\ x\ \vec{B})\cdot(\vec{C}\ x\ \vec{D})=(\vec{A}\cdot\vec{C})(\vec{B}\cdot\vec{D})-(\vec{A}\cdot\vec{D})(\vec{B}\cdot\vec{C})$$

And

$$\vec{A}\ x\ (\vec{B}\ x\ (\vec{C}\ x\ \vec{D}))=\vec{B}(\vec{A}\cdot(\vec{C}\ x\ \vec{D}))-(\vec{A}\cdot\vec{B})(\vec{C}\ x\ \vec{D})$$

To answer the question, look specifically at the "BAC-CAB" rule: let vector A be a pseudo vector, and vector B and C are simple vectors. Then under the inversion of coordinates, A is invariant where-as B and C are not. Thus, the results of this is a vector which is invariant under inversion. **So we conclude that the results of the cross product of two pseudovectors is also a pseudovector since it is invariant under the inversion transformation.**

So clearly the vector formed by the cross product of two vectors, each of which undergoes and inversion, does not transform according to the rule developed for vectors above. For this reason, the vector formed by the cross product of two vectors is actually called a pseudovector because it does not behave like a vector under the inversion transformation. It is said that a pseudovector is invariant under an inversion transformation.

(d) The scalar triple product of three vectors $\vec{A},\vec{B},\vec{C}$ is given by: $\vec{A}\cdot(\vec{B}\ x\ \vec{c})$. The scalar triple product can be written in the three equivalent forms:

$$\vec{A}\cdot(\vec{B}\ x\ \vec{C})=\vec{B}\cdot(\vec{C}\ x\ \vec{A})=\vec{C}\cdot(\vec{A}\ x\ \vec{B})$$

The components of such a vector would (using the third form) be given by:

$$\vec{C}\cdot(\vec{A}\ x\ \vec{B})=C_x(A_y B_z-B_y A_z)\hat{x}+C_y(B_x A_z-A_x B_z)\hat{y}+C_z(A_x B_y-B_x A_y)\hat{z}$$

Clearly then, the dot of a vector with a pseudovector produces a scalar quantity which is not invariant under an inversion transformation. This is different from what the normal dot product between two vectors would do:

$$\vec{A}\cdot\vec{B}=A_x B_x+A_y B_y+A_z B_z$$

This scalar is invariant under an inversion transformation where-as the pseudoscalar formed as the result of the scalar triple product operation is not invariant under an inversion transformation. Magnetic flux is just such a physical quantity and this in part accounts for its unusual properties with regard to measurement after rotation of superconducting coils. **We can say that a pseudoscalar is not invariant under an inversion transformation.**