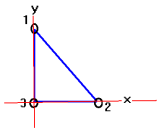


**Concepts:** quantization of charge, law of charges, Coulomb's Law, Superposition of electrostatic forces

(1) A metal sphere has a charge of  $+8.0\mu\text{C}$ . What is the net charge after  $6.0 \times 10^{13}$  electrons have been placed on it?

(2) Two very small spheres are initially neutral and their centers are separated by a distance of 0.50 m. Suppose that  $3.0 \times 10^{13}$  electrons are removed from one sphere and placed on the other. (a) What is the magnitude of the electrostatic force that acts on each sphere? (b) Is the force attractive or repulsive? Why?

(3) In a vacuum, two particles have charges of  $q_1$  and  $q_2$ , where  $q_1 = +3.5\mu\text{C}$ . They are separated by a distance of 0.26 m, and particle 1 experiences an attractive force of 3.4 N. What is  $q_2$  (magnitude and sign)?



(4) An isosceles right triangle has sides of  $L = 0.15$  m.

Charges  $(q_1, q_2, q_3) = (-9, +8, +3)\mu\text{C}$  are located at the corners of the triangle as shown. Find the magnitude of the net electrostatic force exerted on the  $-9.0\mu\text{C}$  charge.

(5) Three charges are fixed to a Cartesian coordinate system. A charge of  $+18\mu\text{C}$  is on the y axis at  $y = +3.0$  m. A charge of  $-12\mu\text{C}$  is at the origin. Lastly, a charge of  $+45\mu\text{C}$  is on the x axis at  $x = +3.0$  m. Determine the magnitude and direction of the net electrostatic force on the charge at  $x = +3.0$  m. Specify the direction relative to the -x axis.

### Introduction to electrostatics

Here, we begin talking about electrostatics with a pretty simple observation, namely that there are 2 types of charges (+ and -) and that the law of charges is obeyed. **The law of charges states that like charges repel while unlike charges attract.**

Charges are measured in units of Coulombs. The smallest charge (other than 0) is  

$$e = -1.602 \times 10^{-19} \text{ C}$$

which is the charge on the electron. We also have the charge on the proton which is the same as the charge on the electron but opposite in sign.

All other charges are given by integral multiples of the fundamental charge:

$$q = Ne \quad (\text{or } Q = Ne)$$

where N (an integer) is the number of electrons (or, protons if q is positive).

### Coulomb's Law

The amount of force between two charges, like or unlike, is governed by Coulomb's law. Suppose you have a charge  $q_1$  and a second charge  $q_2$  located at a distance  $r$  from each other. The force of attraction or repulsion between on charge 2 due to charge 1 is given by Coulomb's law:

$$\vec{F}_{21} = k \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12} \quad \text{or} \quad \vec{F}_{pi} = k \frac{q_i q_p}{r_{ip}^2} \hat{r}_{ip}$$

where  $k$  is called "Coulomb's Constant" ...  $k = 8.99 \times 10^9 \text{ Nm}^2/\text{C}^2$  and  $r$  is the distance of separation between the two charges. **It is useful to note that the indices here are reversed across the equal sign.**

I am going to try to always use the following notations:

$\vec{F}_{21}$  will mean the force on body 2 due to body 1.

Here,  $r$  is the distance between the two charges.

$\hat{r}_{12}$  is the unit vector along the radial direction which points from charge 1 to charge 2.

**Important notes on the unit and other vectors we'll be using.  
 Make 100% sure you do not go further without knowing this!**

Suppose charge  $i$  is located at  $\vec{r}_i = (x_i, y_i, z_i)$ . We are going to need to know the vector that points from  $i$  to a point  $p$  in space given by the coordinates:

$$\vec{r}_p = (x_p, y_p, z_p)$$

### How do we find the unit vectors and other needed items?

(1) Write you charge and point coordinates as vectors:

$$\vec{r}_i = x_i \hat{x} + y_i \hat{y} + z_i \hat{z}; \quad \vec{r}_p = x_p \hat{x} + y_p \hat{y} + z_p \hat{z}$$

(2) The vector pointing from charge  $i$  to point  $p$  is given by:

$$\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = [(x_p - x_i) \hat{x} + (y_p - y_i) \hat{y} + (z_p - z_i) \hat{z}]$$

(3) The unit vector pointing from charge  $i$  to point  $p$  is given by:

$$\hat{r}_{ip} = \frac{\vec{r}_{ip}}{|\vec{r}_{ip}|} = \frac{(x_p - x_i) \hat{x} + (y_p - y_i) \hat{y} + (z_p - z_i) \hat{z}}{\sqrt{(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2}}$$

Here are a couple of things that you will find useful

$$\frac{1}{|\vec{r}_{ip}|} \hat{r}_{ip} = \frac{\hat{r}_{ip}}{|\vec{r}_p - \vec{r}_i|} = \frac{1}{|\vec{r}_{ip}|} \frac{1}{|\vec{r}_{ip}|} \vec{r}_{ip} = \frac{(x_p - x_i) \hat{x} + (y_p - y_i) \hat{y} + (z_p - z_i) \hat{z}}{(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2}$$

$$\frac{1}{|\vec{r}_{ip}|^2} \hat{r}_{ip} = \frac{1}{|\vec{r}_{ip}|^2} \frac{1}{|\vec{r}_{ip}|} \vec{r}_{ip} = \frac{(x_p - x_i)\hat{x} + (y_p - y_i)\hat{y} + (z_p - z_i)\hat{z}}{[(x_p - x_i)^2 + (y_p - y_i)^2 + (z_p - z_i)^2]^{3/2}}$$

### Superposition of electrostatic forces

The electrostatic forces from several discrete charge distributions which are spatially separated superimpose vectorially. This means that the force on a charge “p” located at position  $\vec{r}_p$  due to n charges (numbered 1, 2, 3, ..., n) located at positions  $\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n$  is given by:

$$\vec{F}_p = \sum_{\substack{i=1 \\ i \neq p}}^n \vec{F}_{pi} = \sum_{\substack{i=1 \\ i \neq p}}^n k \frac{q_i q_p}{|\vec{r}_{ip}|^2} \hat{r}_{ip}$$

This probably looks a little bit scary but with correct symmetries and lower dimensions, the problem is reduced significantly. Do note, however, that almost all of electrostatics is contained in this one equation! In one dimension, this is pretty easy to work with normally. I am going to show you in succeeding lectures, and in this lecture, the technical details behind application of this to discrete charge distributions. I recommend the technical method over application of symmetries that you see in problems. But it is tedious and you need to do it with a clear mind and remain focused throughout your calculation.

One thing that you’ll need to know here is how a metal conductor behaves under electrostatics ... the charges placed on a metal conductor all spread out over the outer edge of the conductor. They do this in such a way as to minimize the electrostatic forces on any one charge. So long as the symmetry is spherical, outside the system the charge acts as if it is concentrated at the center of the sphere. I’ll show you this later but for now, you’ll need to accept it.

(1) A metal sphere has a charge of  $+8.0\mu\text{C}$ . What is the net charge after  $6.0 \times 10^{13}$  electrons have been placed on it?

Solution:

$$\begin{aligned} Q &= q_1 + Ne = +8.0 \times 10^{-6} \text{C} + 6.0 \times 10^{13} (-1.602 \times 10^{-19} \text{C}) \\ &= 8.0 \times 10^{-6} - 9.612 \times 10^{-6} = -1.612 \times 10^{-6} \text{C} = -1.612 \mu\text{C} \end{aligned}$$

(2) Two very small spheres are initially neutral and their centers are separated by a distance of 0.50 m. Suppose that  $3.0 \times 10^{13}$  electrons are removed from one sphere and placed on the other. (a) What is the magnitude of the electrostatic force that acts on each sphere? (b) Is the force attractive or repulsive? Why?

Solution:

(a)

This is an application of Coulomb's law:  
here,

$$\begin{aligned} q_1 &= +3.0 \times 10^{13} (1.602 \times 10^{-19} \text{C}) = +4.806 \times 10^{-6} \text{C} \\ q_2 &= -3.0 \times 10^{13} (1.602 \times 10^{-19} \text{C}) = -4.806 \times 10^{-6} \text{C} \end{aligned}$$

$$r = 0.5 \text{m}$$

So,

$$|\vec{F}| = k \frac{q_1 q_2}{r^2} = 8.99 \times 10^9 \frac{(4.806 \times 10^{-6})^2}{(0.5)^2} = 0.831 \text{N}$$

By Newton's law, the magnitude of the force on each sphere is the same.

(b) The force is attractive by the law of charges.

(3) In a vacuum, two particles have charges of  $q_1$  and  $q_2$ , where  $q_1 = +3.5\mu\text{C}$ . They are separated by a distance of 0.26 m, and particle 1 experiences an attractive force of 3.4 N. What is  $q_2$  (magnitude and sign)?

Solution:

Let's do the easier part first ...

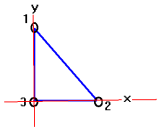
If the force is attractive, since  $q_1$  is positive,  $q_2$  must be negative.

Now, let's find out how big  $q_2$  is:

$$|\vec{F}| = \left| k \frac{q_1 q_2}{r^2} \right| = 3.4 \text{ N} \Rightarrow |q_2| = \frac{3.4 r^2}{k q_1} = \frac{3.4 (0.26)^2}{8.99 \times 10^9 (3.5 \times 10^{-6})} = \frac{0.22984}{31465} = 7.3 \times 10^{-6} \text{ C} = 7.3 \mu\text{C}$$

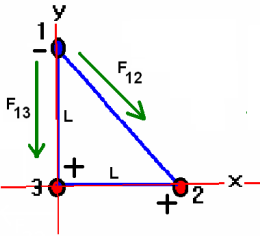
so  $q_2 = -7.3\mu\text{C}$ .

I have reproduced this problem on a spreadsheet.



(4) An isosceles right triangle has sides of  $L=0.15$  m.

Charges  $(q_1, q_2, q_3) = (-9, +8, +3) \mu\text{C}$  are located at the corners of the triangle as shown. Find the magnitude of the net electrostatic force exerted on the  $-9.0 \mu\text{C}$  charge.



Solution: I am going to show all the technical steps needed. There are simpler ways to do this, but if you follow this procedure, you will always arrive at the correct solution without having to use trigonometry or apply symmetry arguments. This may or may not be a good thing to show. Let's place a Cartesian coordinate system on this problem as shown. Then, (using  $L$  as the length):

$$\vec{r}_1 = 0\hat{x} + L\hat{y} + 0\hat{z}; \vec{r}_2 = L\hat{x} + 0\hat{y} + 0\hat{z}; \vec{r}_3 = 0\hat{x} + 0\hat{y} + 0\hat{z}$$

It is now easy to apply Coulomb's law to calculate the force on charge 1:

$$\vec{F}_p = \vec{F}_1 = \sum_{i=2}^3 \vec{F}_{1i} = \vec{F}_{12} + \vec{F}_{13} = \sum_{i=2}^3 k \frac{q_1 q_i}{|\vec{r}_{1i}|^2} \hat{r}_{1i} = \sum_{i=2}^3 k \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \hat{r}_{12} + k \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|^2} \hat{r}_{13}$$

Let's calculate the vector quantities needed.

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = (0-L)\hat{i} + (L-0)\hat{j} + 0\hat{k} = -L\hat{i} + L\hat{j} \Rightarrow |\vec{r}_1 - \vec{r}_2| = L\sqrt{2}; \hat{r}_{21} = \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \frac{-L\hat{i} + L\hat{j}}{L\sqrt{2}} = \frac{-\hat{i} + \hat{j}}{\sqrt{2}}$$

$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = (0-0)\hat{i} + (L-0)\hat{j} + (0-0)\hat{k} = L\hat{j}; \hat{r}_{31} = \frac{\vec{r}_1 - \vec{r}_3}{|\vec{r}_1 - \vec{r}_3|} = \frac{L\hat{j}}{L} = \hat{j}$$

We can now write the force using  $(q_1, q_2, q_3) = (-9, +8, +3) \mu\text{C}$

$$\vec{F}_1 = k(-9) \left[ \frac{8}{(L\sqrt{2})^2} \frac{-\hat{i} + \hat{j}}{\sqrt{2}} + \frac{3}{(L)^2} \hat{j} \right] = \frac{-9k}{L^2} \left[ \frac{-4}{\sqrt{2}} \hat{i} + \left( \frac{4}{\sqrt{2}} + 3 \right) \hat{j} \right] \mu\text{C} = \frac{k}{L^2} [25.4558 \hat{i} - 52.4558 \hat{j}] \mu\text{C}$$

$$\Rightarrow \vec{F}_1 = \frac{0.2288}{L^2} \hat{i} - \frac{0.4716}{L^2} \hat{j} \text{ N} = 10.17 \hat{i} - 20.96 \hat{j} \text{ N}$$

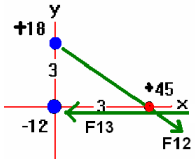
$$\Rightarrow |\vec{F}_1| = \sqrt{\vec{F}_1 \cdot \vec{F}_1} = \sqrt{10.17^2 + 20.96^2} = 23.30 \text{ N}$$

$$\tan \theta = \frac{\vec{F}_1 \cdot \hat{j}}{\vec{F}_1 \cdot \hat{i}} = -\frac{20.9}{10.2} \Rightarrow \theta = -64.12^\circ$$

The force is pointing 64.12 degrees below the x-axis. Note that you need to be a bit careful to not get your angle off by 180 degrees when you calculate these directions.

I've produced this problem in a spreadsheet also.

(5) Three charges are fixed to an xy coordinate system. A charge of  $+18 \mu\text{C}$  is on the y axis at  $y=+3.0 \text{ m}$ . A charge of  $-12 \mu\text{C}$  is at the origin. Lastly, a charge of  $+45 \mu\text{C}$  is on the x axis at  $x=+3.0 \text{ m}$ . Determine the magnitude and direction of the net electrostatic force on the charge at  $x=+3.0 \text{ m}$ . Specify the direction relative to the  $+x$  axis.



Solution:

The picture looks like this:

I want to do this again with the “technical but tedious” approach that I am teaching you. I think it is going to be helpful if I say that you need to put all your given data into a tabular form. You really ought to do this since it will save lots of time and reduce mistakes. I suppose it needs to look something

like this:

Charge ( $\mu\text{C}$ )	x	y	z
1:+45	+3	0	0
2:+18	0	+3	0
3:-12	0	0	0

Now we’re going to need to calculate the various unit vectors and distances. As you can see this is very similar to the last problem.

$$\vec{r}_1 = 3\hat{i} + 0\hat{j}; \vec{r}_2 = 0\hat{i} + 3\hat{j}; \vec{r}_3 = 0\hat{i} + 0\hat{j}$$

$$\vec{F}_p = \vec{F}_1 = \sum_{i=2}^3 \vec{F}_{1i} = \vec{F}_{12} + \vec{F}_{13} = \sum_{i=2}^3 k \frac{q_1 q_i}{|\vec{r}_{ip}|^2} \hat{r}_{i1} = \sum_{i=2}^3 k \frac{q_1 q_i}{|\vec{r}_1 - \vec{r}_i|^2} \hat{r}_{i1} = k \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \hat{r}_{21} + k \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|^2} \hat{r}_{31}$$

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = (3\hat{i} + 0\hat{j}) - (0\hat{i} + 3\hat{j}) = 3\hat{i} - 3\hat{j}; |\vec{r}_{21}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}; \hat{r}_{21} = \frac{3\hat{i} - 3\hat{j}}{\sqrt{3^2 + 3^2}} = \frac{\hat{i} - \hat{j}}{\sqrt{2}}$$

$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = (3 - 0)\hat{i} + (0 - 0)\hat{j} + 0\hat{k} = 3\hat{i}; |\vec{r}_{31}| = \sqrt{3^2} = 3; \hat{r}_{31} = \frac{3\hat{i}}{3} = \hat{i}$$

We are now ready to form the force on charge 1:

$$\vec{F}_1 = k q_1 \left[ \frac{18}{9 \times 2} \frac{\hat{i} - \hat{j}}{\sqrt{2}} + \frac{-12}{9} \hat{i} \right] \mu = k q_1 \left[ \left( \frac{1}{\sqrt{2}} - \frac{12}{9} \right) \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right] \mu = k(45) [-.626\hat{i} - 0.707\hat{j}] \mu$$

$$\vec{F}_1 = [-.253\hat{i} - 0.287\hat{j}] \text{N}$$

$$|\vec{F}_1| = \sqrt{.253^2 + .287^2} = 0.383 \text{N}$$

$$\tan \theta = \frac{F_y}{F_x} = \frac{-.287}{-.253} = 1.134 \Rightarrow \theta = 48.6^\circ \text{ or } \theta = 180 + 48.6^\circ = 228.6^\circ$$

The correct answer is 228.6

One way you can speed up your calculations when you place numbers into your calculations is this: if all your charges are given in micro-coulombs then place them in without multiplying by  $10^{-6}$  and multiply by  $10^{-12}$  at then end. Also note that  $k_\mu = 8990$  in SI units. To get the second  $\mu$  out, divide your force by  $10^{-6}$  at the end.

I’ve produced this problem in a spreadsheet also.

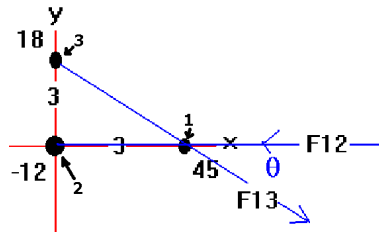
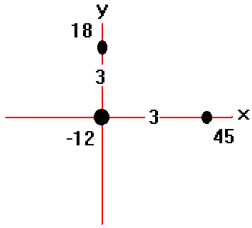
**Below is another version of the same approach for this problem.**

(5) Three charges are fixed to a Cartesian coordinate system. A charge of  $+18 \mu\text{C}$  is on the  $y$  axis at  $y=+3.0$  m. A charge of  $-12 \mu\text{C}$  is at the origin. Lastly, a charge of  $+45 \mu\text{C}$  is on the  $x$  axis at  $x=+3.0$  m. Determine the magnitude and direction of the net electrostatic force on the charge at  $x=+3.0$  m. Specify the direction relative to the  $-x$  axis.

Solution:

The picture looks like this:

To find the forces, you need to resolve components of the vectors as shown below: (note  $\theta=45^\circ$ ).



The force is then given by:

$$\vec{F}_p = \sum_{\substack{i=1 \\ i \neq p}}^n \vec{F}_{pi} = \sum_{\substack{i=1 \\ i \neq p}}^n k \frac{q_i q_p}{|\vec{r}_{ip}|^2} \hat{r}_{ip} = k \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} (-\hat{x}) + k \frac{q_1 q_3}{|\vec{r}_1 - \vec{r}_3|^2} (+\cos(\theta)\hat{x} - \sin(\theta)\hat{y})$$

It is easy to find the angle using trigonometry: by symmetry it must be  $45^\circ$ . The rest of the solution proceeds as above. The first method, however, is preferred by me since it does not require you to use symmetry considerations.

**If, however, the point of interest is at the origin** (i.e. calculate the force on a charge located at the origin), then the vector calculations do simplify since  $\vec{r}_p = 0\hat{x} + 0\hat{y} + 0\hat{z}$  :

$$\vec{r}_{ip} = \vec{r}_p - \vec{r}_i = -\vec{r}_i : |\vec{r}_{ip}| = |\vec{r}_i| : \hat{r}_{ip} = \frac{-\vec{r}_i}{|\vec{r}_i|}$$

But this only applies in the special case that the point where you want to find the force on is located at the origin which it is not, in general.

**Summary: step by step instructions:**

**a:** Write vectors pointing to each charge:

$$\vec{r}_1 = 3\hat{i} + 0\hat{j} : \vec{r}_2 = 0\hat{i} + 3\hat{j} : \vec{r}_3 = 0\hat{i} + 0\hat{j}$$

**b:** Form the vectors pointing towards the charge of interest (charge 1 in this case):

$$\vec{r}_{21} = \vec{r}_1 - \vec{r}_2 = (3\hat{i} + 0\hat{j}) - (0\hat{i} + 3\hat{j}) = 3\hat{i} - 3\hat{j} : |\vec{r}_{21}| = \sqrt{3^2 + 3^2} = 3\sqrt{2}$$

$$\vec{r}_{31} = \vec{r}_1 - \vec{r}_3 = (3 - 0)\hat{i} + (0 - 0)\hat{j} + 0\hat{k} = 3\hat{i} : |\vec{r}_{31}| = \sqrt{3^2} = 3$$

**c:** Calculate the unit vectors.

$$\hat{r}_{21} = \frac{3\hat{i} - 3\hat{j}}{\sqrt{3^2 + 3^2}} = \frac{\hat{i} - \hat{j}}{\sqrt{2}} \quad \hat{r}_{31} = \frac{3\hat{i}}{3} = \hat{i}$$

**d:** put these results into each term of the Coulomb's force expression and calculate it:

$$\vec{F}_1 = k q_1 \left[ \frac{18}{9 \times 2} \frac{\hat{i} - \hat{j}}{\sqrt{2}} + \frac{-12}{9} \hat{i} \right] \mu = k q_1 \left[ \left( \frac{1}{\sqrt{2}} - \frac{12}{9} \right) \hat{i} - \frac{1}{\sqrt{2}} \hat{j} \right] \mu = k(45) [-6.26\hat{i} - 0.707\hat{j}] \mu$$

$$\vec{F}_1 = [-.253\hat{i} - 0.287\hat{j}] \text{N}$$

Find the magnitude and angle:

$$|\vec{F}_1| = \sqrt{.253^2 + .287^2} = 0.383 \text{N} : \phi = \tan^{-1} \left( \frac{-0.287}{-0.253} \right) = 48.6^\circ \vee \phi = 48.6 + 180^\circ = 228.6^\circ (\text{correct})$$