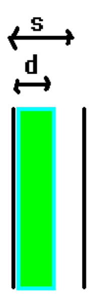
2016	Physics **220**: Worksheet5	Name ____________

Concepts: Capacitance, dielectric constant, resistance, series/parallel combinations

(1) A coaxial cable consists of an insulator of inner radius a with charge/length $+\lambda$ and an outer insulating cylinder of radius b with charge/length $-\lambda$. (a) Find the electric field everywhere in space. (b) Find the potential difference between the inner and outer edges. (c) Find the capacitance of the system if the cable has a total length L.

(2) A dielectric material is characterized by a dielectric constant κ and, if linear, when inserted into a capacitor, will produce a capacitance $C=\kappa C_{geo}$ where C_{geo} is called the "geometric capacitance" and is the capacitance of the capacitor without material inserted between the plates. As a range, a vacuum has a dielectric constant of 1.000000 while water has a dielectric constant of 80.



(a) Suppose you measured a capacitance of 100 pf in a capacitor which was empty and you measured a capacitance of 300 pf when the capacitor was immersed in fluid. What is the dielectric constant of the fluid?

(b) Suppose a slab of this material of thickness d and area A was inserted inside of a parallel plate capacitor with plates of area A and thickness s (s>d). Find the capacitance of this system.

(3) Obtain the addition formulas for capacitors in series and in parallel.

(4) Obtain the equivalent resistance for resistors in series and in parallel.

(5) Suppose Chad places two materials in series. Both of the materials have the same length (L) and the same cross sectional area A. The first material has a resistivity ρ_1 while the second material has a resistivity ρ_2. Find the resistance of the series combination

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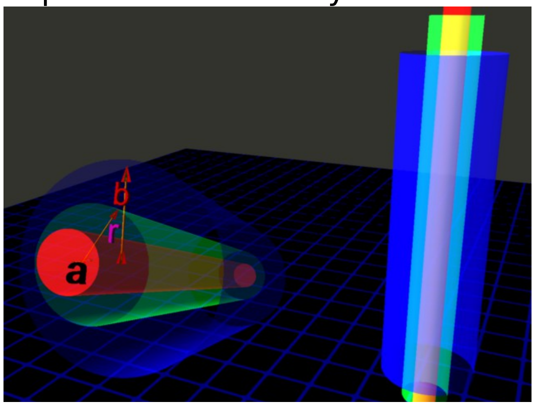
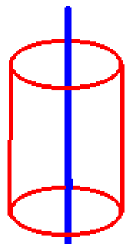
(1) A coaxial cable consists of an insulating cylinder of inner radius a with charge/length $+\lambda$ (on its surface) and an outer cylinder of radius b with charge/length $-\lambda$. (a) Find the electric field everywhere in space. (b) Find the potential difference between the inner and outer edges. (c) Find the capacitance of the system if the cable has a total length h.



We know how to calculate the electric field from a long wire:



Choose the Gaussian cylinder as shown of radius r and height h. In the region between the coaxial cables, we then have:

$$\Phi_E = \vec{E} \cdot \vec{A} = E(2\pi s h)$$

The enclosed charge is given by: $q_{enc} = \lambda h$ (we assume the single wire has zero radius here). We can equate the two expressions via Gauss's law:

$$E(2\pi s h) = \frac{\lambda h}{\epsilon_0} \Rightarrow \vec{E} = \frac{\lambda}{2\pi s \epsilon_0}\hat{s}$$

Now let me expand this to include the case of a single cylinder of radius a. In this case, the charge (if it is a conducting wire) will be all located on the surface of the wire.

If s<a, then no net charge is enclosed. This means:

$$\vec{E}_{s<a} = \vec{0}$$

If you are outside the cylinder of radius a, then you are enclosing a net charge. As before, we then have the electric field is given by:

$$\vec{E}_{s>a} = \frac{\lambda}{2\pi s \epsilon_0}\hat{s}$$

We can now easily expand this to include the second cylinder. If you are at a radius s>b, then no net charge is enclosed. Thus, outside the coaxial cable, the net electric field is given by:

$$\vec{E}_{s>b} = \vec{0}$$

So we can now combine the solutions:

$$\vec{E} = \begin{matrix} \vec{0} & (s<a) \\ \frac{\lambda}{2\pi s \epsilon_0}\hat{s} & (a<s<b) \\ \vec{0} & (s>b) \end{matrix}$$

Now we need to calculate the potential difference between the two cylinders.

Non-Calculus:

It is a bit more difficult for non-calculus students to find the potential difference between the inner and outer conductors. However, here it is:

$$|\Delta V| = |\vec{E}\cdot\Delta\vec{r}| = \left|\frac{\lambda}{2\pi\epsilon_0}\frac{\Delta s}{s}\right| \approx \left|\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)\right|$$

You may remember our often-used approximation that side-steps lots of calculus which was:

$$\left[\frac{\Delta x}{x}\right]_a^b \approx \ln\left(\frac{b}{a}\right)$$

Once you have the potential difference, you next need to calculate the total charge separation. This is given by:

$$W = \lambda h$$

We can find the capacitance from:

$$C \equiv \left|\frac{Q}{\Delta V}\right| = \left|\frac{\lambda h}{\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)}\right| = \frac{2\pi\epsilon_0 h}{\ln\left(\frac{b}{a}\right)} \Rightarrow \frac{C}{h} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

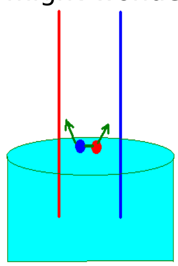
If the cable has a total length h, then

$$C = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)} h$$

Note: I have purposely used h to represent the length rather than L. We will use L later to represent inductance.

The following is an advanced application: you won't see this on a test.

Here is an interesting variation on this problem. Suppose this coaxial cable is inserted into a dielectric fluid with a dielectric constant κ. Depending upon the nature of the fluid, it may either raise or lower within the space of the cable. You might wonder why and also to what height it will raise. Let's suppose the potential difference is strictly maintained to be V. Charges above the fluid will exert a force on the fluid. If the fluid is polar (such as water) then the force exerted by the charges is initially as shown in the sketch. This force will enable the fluid to raise up into the area between the plates. How high the fluid will raise is possible to calculate. When a column of fluid raises through a distance h_1, the change in gravitational potential energy is given by the average height to which a disk of height h_1 is raised:



$$\Delta U = Mgh_{1_{average}} = \left[\rho_{fluid}\left(\pi\left(b^2-a^2\right)\right)\right]gh_{1_{average}} = \frac{1}{2}\rho_{fluid}\pi\left(b^2-a^2\right)gh_1^2$$

The work required to raise this fluid came from the electrostatic potential difference, which was maintained to be a constant. This means that addition charge had to be separated. We can find out how much additional charge needed to be separated pretty easily. We have essentially two capacitors in parallel when the system has finally achieved equilibrium. Later in this worksheet you will find that capacitances in parallel add to give an equivalent capacitance. This then tells up what additional charge was required to be separated:

$$C_{parallel} \Rightarrow C_{eq} = C_1 + C_2 = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\left[h - h_1 + \kappa h_1\right] = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\left[h + h_1(\kappa - 1)\right]$$

The amount of additional charge which needed to be separated is then:

$$\Delta Q = Q_1 - Q = V\left[\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\right]\left(h + h_1(\kappa-1) - h\right) = V\left[\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\right]\left(h_1(\kappa-1)\right)$$

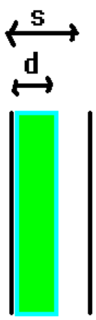
The additional work required to produce this charge is then given by:

$$W = V\,\Delta Q = V^2\left[\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\right]\left(h_1(\kappa-1)\right)$$

This work went into raising the dielectric fluid into the system. Thus:

$$V^2\left[\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}\right]\left(h_1(\kappa-1)\right) = \frac{1}{2}\rho_{fluid}\pi\left(b^2-a^2\right)g\,h_1^2 \Rightarrow h_1 = \frac{V^2\left[\frac{2\pi\epsilon_0}{\ln(b/a)}\right](\kappa-1)}{\frac{1}{2}\rho_{fluid}\pi\left(b^2-a^2\right)g}$$

(2) A dielectric material is characterized by a dielectric constant κ and, if linear, when inserted into a capacitor, will produce a capacitance $C=\kappa C_{geo}$ where C_{geo} is called the “geometric capacitance” and is the capacitance of the capacitor without material inserted between the plates. As a range, a vacuum has a dielectric constant of 1.000000 while water has a dielectric constant of 80.



(a) Suppose you measured a capacitance of 100 pf in a capacitor which was empty and you measured a capacitance of 300 pf when the capacitor was immersed in fluid. What is the dielectric constant of the fluid?

(b) Suppose a slab of this material of thickness d and area A was inserted inside of a parallel plate capacitor with plates of area A and thickness s (s>d). Find the capacitance of this system.

(a) We find the dielectric constant from: $\kappa=\frac{C}{C_{geo}}$. Now, we compare the two measurements of capacitance to find:

$$\kappa=\frac{300\,\text{pf}}{100\,\text{pf}}=3$$

(b) To answer the second part, we will need to use a result which will be coming, namely that for n capacitors in series, the total capacitance is given by:

$$\frac{1}{C_{equivalent}}=\sum_{i=1}^{n}\frac{1}{C_i}$$

Since we’re inserting this material into a parallel plate capacitor, and the material is in the form of a slab, we can find the capacitance of each individual part of the system:

We had, for a parallel plate capacitor:

$$C=\epsilon_0\frac{A}{d}$$

The capacitance of each individual part is this:

I(a): slab region without the slab: $C=\epsilon_0\frac{A}{d}=C_{geo}$

I(b): the slab region with the slab: $C=\kappa C_{geo}=\kappa\epsilon_0\frac{A}{d}\equiv C_1$

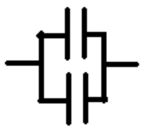
II: The region with out the slab. $C=\epsilon_0\frac{A}{|s-d|}\equiv C_2$

Now we can find the equivalent capacitance:

$$\frac{1}{C_{eq}}=\frac{1}{C_1}+\frac{1}{C_2}=\frac{d}{\kappa\epsilon_0 A}+\frac{s-d}{\epsilon_0 A}=\frac{s}{\epsilon_0 A}\left[\frac{d}{\kappa s}+\frac{\kappa(s-d)}{\kappa s}\right]=\frac{s}{\epsilon_0 A}\left[\frac{d+\kappa s-\kappa d}{\kappa s}\right]\Rightarrow C_{eq}=\kappa\frac{\epsilon_0 A}{d+\kappa(s-d)}$$

You can verify that if d=s the correct capacitance is obtained. You can also verify that if d=0, the correct geometrical capacitance results.

(3) Obtain the addition formulas for capacitors in series and in parallel.



Solution:

PARALLEL

If two capacitors are in parallel, they have a common potential difference across their plates but the charge stored on each capacitor may be different. It is a common potential difference because the wires connecting the two capacitors are equipotential surfaces. Thus, we have:

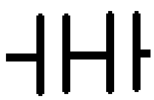
$$\Delta V=\frac{Q_1}{C_1}=\frac{Q_2}{C_2}$$

We have that the total charge stored is given by:

$$Q=Q_1+Q_2=\left(C_1+C_2\right)\Delta V=C_{eq}\,\Delta V$$

Thus, for n capacitors in parallel, an equivalent capacitance is given by:

$$C_{\text{equivalent}}^{\text{parallel}}=\sum_{i=1}^{n} C_i$$



SERIES

If two capacitors are in series, the will in general have a different potential “drop” across their plates. However, the total charge separation on each capacitor will be the same. Because charge in between the two outer plates is the same, you can think of actually removing the center piece. In that situation, from the calculation that gave the capacitance of the parallel plate capacitor, you could derive the result below. However, using the condition that the charge is the same across both capacitors, we have: The equivalent capacitance is then:

$$\Delta V=\Delta V_i+\Delta V_2=\frac{Q}{C_1}+\frac{Q}{C_1}=Q\left(\frac{1}{C_1}+\frac{1}{C_2}\right)=\frac{Q}{C_{\text{equivalent}}^{\text{series}}}$$

This shows then that for n capacitors in series, the equivalent capacitance is given by:

$$\frac{1}{C_{\text{equivalent}}^{\text{series}}}=\sum_{i=1}^{n}\frac{1}{C_i}$$

Not all circuits can be expressed as a simple combination of series or parallel capacitors. With those circuits, you will require more powerful techniques in order to do proper analysis.

When charges start flowing, one measures a current. For the present time, we are going to consider a DC current which means that the charges are flowing at a constant rate in a constant direction. The SI unit which measures current is the Ampere and the current is defined (for now) as:

$$I=\frac{\Delta Q}{\Delta t} \quad .$$

It is easy to see from this that 1 Amp= 1 C/1 s in the SI system of units. Later, after we cover magnetic fields, we will have a much better way to define current since, in reality, the electrons are not actually flowing very much but it is the electrostatic force which gets transmitted.

While I am talking about current, I need to mention that we regard our current as the flow of positive charges (the conventional current) due to a bad guess some time ago. In reality, however, what does flow are the electrons (but in the opposite direction).

There are entities which oppose the flow of a current called resistors. It is observed experimentally that in the vast majority of resistances, Ohm's law is obeyed. Ohm's law states:

$$V=IR$$

In fact, what you regard as Ohm's law, you probably are more correct as regarding to be the definition of resistance:

$$R\equiv\frac{V}{I} \quad ,$$

and Ohm's law would be a statement that for a given material, the ratio of V to I is a linear relationship. Now, in fact the resistance is almost always measured as being the ratio of V to I but not all materials show a linear relationship between V and I. These materials are called "non-ohmatic" resistances.

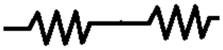
The SI system of resistance is the Ohm Ω and it is easy to see that in the SI system, the units for the Ohm are V/A or (J/C)/(C/s)=Js/C^2.

I want to show you in the next problem how resistance in series and parallel behave. Before that, however, I want to show you how to calculate the resistance of a material. To do so, we define the resistivity of a material and give it the symbol ρ. It has the same symbol as volume charge density but the meaning is quite different. In short, imagine you have a cylinder of material with a cross sectional area A and a length L. The resistance of this material is then given by:

$$R=\rho\frac{L}{A} \quad .$$

In the SI system, resistivity has units of Ωm. Metals have a resistivity of about 10^{-8} Ωm while glass has a resistivity on the order of $10^{10}\Omega$m. However, to be truthful, resistivity needs to be specified with regard to the temperature range of use. Glass at room temperature is an insulator but if you heat it up, it will become a conductor shortly before it melts. Metals, on the other hand, become less conducting as the temperature is increased. This is a primary difference between glassy materials and metals.

(4) Obtain the equivalent resistance for resistors in series and in parallel.



Solution:

SERIES

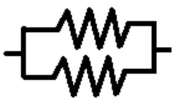
For resistors in series, the current through each resistor is the same but the potential drop across each resistor is in general different. Thus,

$$\Delta V = \Delta V_1 + \Delta V_2 = IR_1 + IR_2 = I(R_1 + R_2) = I R_{equivalent}^{series}$$

Thus, for n resistors in series:

$$R_{equivalent}^{series} = \sum_{i=1}^{n} R_i$$

PARALLEL



For resistors in parallel, the current through each resistor is in general not the same. Instead, the potential drop across each resistor is the same. The reason this is true is each end is connected to the same wire which we regard theoretically as an equipotential surface.

The total current input (or output) is then given by:

$$I = I_1 + I_2 = \frac{V}{R_1} + \frac{V}{R_2} = \frac{V}{R_{equivalent}}$$

It is easy to see from this formulation, then that the equivalent parallel resistance of n resistors is given by:

$$\frac{1}{R_{equivalent}^{parallel}} = \sum_{i=1}^{n} \frac{1}{R_i}$$

Here is an interesting observation that explains why you should not plug too many things into a single circuit:

Suppose you have n identical resistors in parallel, each of resistance R. Then

$$\frac{1}{R_{equivalent}} = \frac{n}{R} \Rightarrow R_{eq} = \frac{R}{n}$$

Now suppose your circuit supplies a voltage V to the resistances. The current is given by:

$$I = \frac{V}{R_{eq}} = n\frac{V}{R}$$

As you add more and more devices, n becomes large. The circuit comes closer and closer to a short circuit, which results in an infinite amount of current flowing. Since the power company frowns upon your house taking all their current, the fuse will blow or your house will burn down because the wires have a finite resistance that causes them to act like internal heaters.

(5) Suppose Chad places two materials in series. Both of the materials have the same length (L) and the same cross sectional area A. The first material has a resistivity ρ_1 while the second material has a resistivity ρ_2. Find the resistance of the series combination.

Solution: Each of the resistors has a resistance given by:

$$R_i = \rho_i \frac{L}{A}$$

Since the two resistors are placed in series, the equivalent resistance is given by:

$$E_{eq} = \sum_i R_i = (\rho_1 + \rho_2) \frac{L}{A}$$

If, however, the resistors were placed in parallel, we would have:

$$\frac{1}{R_{eq}} = \sum_i \frac{1}{R_i} = \frac{A}{L}\left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right) \Rightarrow R_{eq} = \frac{L}{A}\left(\frac{\rho_1 \rho_2}{\rho_1 + \rho_2}\right)$$

One final parting note: very many students have difficulty in performing this operation: $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ on calculators and obtaining the correct results for a. This difficulty has been seen in all students at all levels. I recommend that it will be to your great benefit if you make sure that you are able to put numbers in for b and c and solve correctly for a.