**Guided Waves**

Then ... Wave Guides

Assume a perfect conductor so that $\vec{E}^{\parallel}=\vec{0}:B^{\perp}=0$ . B is zero since in a perfect conductor E is zero. Thus the time derivative of E is zero leading to then the curl of B and thus B is zero.

We assume monochromatic waves propagating in the +z direction.

$$\tilde{\vec{E}}(x,y,z,t)=\tilde{\vec{E}}_0(x,y)e^{i(kz-\omega t)}$$
$$\tilde{\vec{B}}(x,y,z,t)=\tilde{\vec{B}}_0(x,y)e^{i(kz-\omega t)}$$

The waves obey Maxwell's equations:

$$\vec{\nabla}\cdot\vec{E}=0 \quad \vec{\nabla}\cdot\vec{B}=0 \quad \vec{\nabla}x\vec{E}=-\frac{\partial\vec{B}}{\partial t} \quad \vec{\nabla}x\vec{B}=\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$$

In general these solutions are not necessarily going to be transverse inside a wave guide. This is going to complicate the problem significantly over the TEM situation. Assume:

$$\tilde{\vec{E}}_0=E_x\hat{x}+E_y\hat{y}+E_z\hat{z}$$
$$\tilde{\vec{B}}_0=B_x\hat{x}+B_y\hat{y}+B_z\hat{z}$$

Note the ~ is left off the individual components here for clarity but the fields may still be complex. These are going to be put through the last two of Maxwell's equations go give:

$$1:\quad \frac{\partial E_y}{\partial x}-\frac{\partial E_x}{\partial y}=i\omega B_z \quad 4:\quad \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}=-\frac{i\omega}{c^2}E_z$$
$$2:\quad \frac{\partial E_z}{\partial y}-ikE_y=i\omega B_x \quad 5:\quad \frac{\partial B_z}{\partial y}-ikB_y=-\frac{i\omega}{c^2}E_x$$
$$3:\quad ikE_x-\frac{\partial E_z}{\partial x}=i\omega B_y \quad 6:\quad ikB_x-\frac{\partial B_z}{\partial x}=-\frac{i\omega}{c^2}E_y$$

It takes some work but (2,3,4,5,6) can be solved for the x and y components:

$$1:\quad E_x=\frac{i}{\left(\frac{\omega}{c}\right)^2-k^2}\left(k\frac{\partial E_z}{\partial x}+\omega\frac{\partial B_z}{\partial y}\right) \quad 2:\quad E_y=\frac{i}{\left(\frac{\omega}{c}\right)^2-k^2}\left(k\frac{\partial E_z}{\partial y}-\omega\frac{\partial B_z}{\partial x}\right)$$
$$3:\quad B_x=\frac{i}{\left(\frac{\omega}{c}\right)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\omega\frac{\partial E_z}{\partial y}\right) \quad 4:\quad B_y=\frac{i}{\left(\frac{\omega}{c}\right)^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\omega\frac{\partial E_z}{\partial x}\right)$$

Rather remarkably, the transverse components are completely specified in terms of the transverse components! This set of 4 equations can be decoupled to become two:

$$1:\quad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0$$
$$2:\quad \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0$$

However, unlike unbound waves, these waves either have $E_z$ =0 (TE) or $B_z$=0 (TM) but not ($E_z$=0 and $B_z$=0) (TEM). Only TEM can propagate in free space; TEM can not propagate in a guide and they must be TE or TM.

**A Rectangular wave guide**

A rectangular wave guide consists of plates running along the z direction, with dimensions a along the x-direction and b along the y-direction. We are assuming here that a is the larger dimension.

We want to solve :
$$1: \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]E_z=0$$
$$2: \left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0$$
for TE waves.

For TE waves, $E_z$=0. this means we need to solve $\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\left(\frac{\omega}{c}\right)^2-k^2\right]B_z=0$ .

This is done by separation of variables. Assume $B_z=X(x)Y(y)$ with separation constants $k_x$ and $k_y$ so that:

$$-k_x^2-k_y^2+\left(\frac{\omega}{c}\right)^2-k^2=0$$

The general solutions are:

$$X(x)=A_1\sin(k_x x)+C_1\cos(k_x x)$$
$$Y(y)=A_2\sin(k_y y)+C_2\cos(k_y y)$$

The boundary conditions require that $B_x$ = 0 and $B_y$=0. (but not $B_z$ ) .This requires: the following: We are working with TE waves, so $E_z$=0.

$$B_x=\frac{i}{\left(\frac{\omega}{c}\right)^2-k^2}\left(k\frac{\partial B_z}{\partial x}-\omega\frac{\partial E_z}{\partial y}\right)\Rightarrow\frac{ik}{\left(\frac{\omega}{c}\right)^2-k^2}\frac{\partial B_z}{\partial x}=0\Rightarrow A_1=0$$
$$B_y=\frac{i}{\left(\frac{\omega}{c}\right)^2-k^2}\left(k\frac{\partial B_z}{\partial y}+\omega\frac{\partial E_z}{\partial x}\right)\Rightarrow\frac{ik}{\left(\frac{\omega}{c}\right)^2-k^2}\frac{\partial B_z}{\partial y}=0\Rightarrow A_2=0$$

$A_1$ and $A_2$ be zero. We then have the solution:

$$B_z(x,y)=C\cos(k_x x)\cos(k_y y)$$

At x=a (and then at y=b) the field must vanish. This gives us the condition on the separation constant:

$$k_x a=n\pi\Rightarrow k_x=\frac{n\pi}{a};n=0,1,2,\ldots$$
$$k_y b=m\pi\Rightarrow k_y=\frac{m\pi}{b};m=0,1,2,\ldots$$

the solution for a particular mode of the magnetic field is then:

$$B_z=B_0\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$$
$$-k_x^2-k_y^2+\left(\frac{\omega}{c}\right)^2-k^2=0\Rightarrow k=\sqrt{\left(\frac{\omega}{c}\right)^2-k_x^2-k_z^2}=\sqrt{\left(\frac{\omega}{c}\right)^2-\left(\frac{n\pi}{a}\right)^2-\left(\frac{m\pi}{b}\right)^2}\ .$$

Not every frequency will propagate. There is a lowest and a highest frequency possible. The lowest frequency that will propagate is when:

$$\omega_{10} = \frac{c\pi}{a}$$ which corresponds to the $TE_{10}$ mode.

The highest frequency which will propagate is given by: the cutoff frequency for a particular mode: $\omega_{mn} \equiv c\pi\sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2}$

In terms of the cutoff frequency, the propagation constant is expressed as:

$$k = \frac{1}{c}\sqrt{\omega^2 - \omega_{mn}^2}$$

The wave velocity here is, interestingly enough, greater than c:

$$v = \frac{\omega}{k} = \frac{c\omega}{\sqrt{\omega^2 - \omega_{mn}^2}} = \frac{c}{\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)^2}}$$

However the speed of energy and information transport is controlled by the group velocity and not the wave velocity. This is given by:

$$v_g = \frac{1}{(dk/d\omega)} = c\sqrt{1 - \left(\frac{\omega_{mn}}{\omega}\right)} < c$$

Which is good: we did not break physics and the universe after all.

A simple example: what is the lowest frequency for a TE mode that will propagate in a wave guide with a=0.1 m and b=0.05 m?

$$\omega_{10} = \frac{c\pi}{a} = \frac{3x10^8}{0.1}\pi = 3x10^9\pi = 2\pi f \Rightarrow f = 1.5 x 10^9 Hz$$

All the higher indices will give higher frequencies. So the $TE_{10}$ mode is the lowest frequency that will propagate.

And just in case you wanted to get your own waveguides sometime in life:

http://www.ebay.com/sch/i.html?_from=R40&_trksid=p2050601.m570.l1313.TR12.TRC2.A0.H0.Xwaveguide.TRS0&_nkw=waveguide&_sacat=0