

9.2.1 EM waves in a vacuum without doing true justice

Recall Maxwell's equations:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

In a vacuum, there are no charges and thus no conventional currents. These become:

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}$$

Take the curl of the last two equations:

And, by the way, we are going to use a vector theorem for curls:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{B}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2}$$

So: $\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \vec{0}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \epsilon_0 \mu_0 \vec{\nabla} \times \left(\frac{\partial \vec{E}}{\partial t} \right) = \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} \quad \text{so:}$$
$$\nabla^2 \vec{B} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2} = 0$$

These represent 3 dimensional (coupled waves) which travel with a speed of:

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c = 3 \times 10^8 \text{ m/s}$$