# Chapter 8: Conservation laws

We have previously from Maxwell's equations obtained the continuity equation:

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla}\cdot\vec{J}$$

For a electric and magnetic fields we have obtained:

$$U_E = \frac{\epsilon_0}{2}\oiiint (\vec{E})^2 d^3 r : U_M = \frac{1}{2\mu_0}\oiiint (\vec{B})^2 d^3 r$$

Thus the total energy stored in and electric and magnetic field is:

$$U_{EM} = \frac{1}{2}\oiiint \left[\epsilon_0 (\vec{E})^2 + \frac{1}{\mu_0}(\vec{B})^2\right] d^3 r$$

This is derivable in another way so we'll follow your author here:

Suppose you have a charge in the presence of and electric field and a magnetic field at a time t. In a later instant, how much work is done by the charges moving around?

Starting with the Lorentz force law, for a particle initially at rest: (so no magnetic force):

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) : \text{Power} = \vec{F}\cdot\vec{v} = \frac{\vec{F}\cdot d\vec{L}}{dt} = q\vec{E}\cdot\vec{v} \Rightarrow \vec{F}\cdot d\vec{L} = q\vec{E}\cdot\vec{v}\, dt$$

but $\quad q = \oiiint \rho\, d^3 r : \vec{J} = \rho\vec{v}$

So the total work done per unit time (the power) is given by:

$$\frac{dW}{dt} = \oiiint \rho\vec{E}\cdot\vec{v}\, d^3 r = \oiiint \vec{E}\cdot\vec{J}\, d^3 r$$

But Ampere's law says: $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$

So $\quad \vec{E}\cdot\vec{J} = \frac{1}{\mu_0}\vec{E}\cdot(\vec{\nabla}\times\vec{B}) - \epsilon_0\vec{E}\cdot\frac{\partial\vec{E}}{\partial t}$

Now use the product rule: $\vec{\nabla}\cdot(\vec{E}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot(\vec{\nabla}\times\vec{B}) \Rightarrow \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{E}) - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$

with Faraday's law: $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

To give: $\quad \vec{E}\cdot(\vec{\nabla}\times\vec{B}) = -\vec{B}\cdot\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B}) = -\frac{1}{2}\frac{\partial B^2}{\partial t} - \vec{\nabla}\cdot(\vec{E}\times\vec{B})$

So:

$$\frac{dW}{dt}=\oiiint\left[-\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t}-\frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t}-\frac{1}{\mu_0}\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)\right]d^3r$$

By Gauss's theorem: $\oiiint\vec{\nabla}\cdot\left(\vec{E}\times\vec{B}\right)d^3r=\oiint\left(\vec{E}\times\vec{B}\right)\cdot d\vec{A}$

We define the Poynting vector as: $\vec{S}\equiv\frac{1}{\mu_0}\left(\vec{E}\times\vec{B}\right)$ .

External work done on the system results in a change in the EM energy of the system and well something else that we're going to show is related to the Poynting vector.

Note the units of the Poynting vector:

$$[\mu_0]=[B]\frac{[\text{Length}]}{[\text{Amps}]}:\frac{1}{[\mu_0]}[B]=\frac{[\text{Amps}]}{[\text{Length}]}:[E]=\frac{[\text{Force}]}{[\text{Coulombs}]}$$

$$[S]=\frac{[\text{Force}]}{[\text{Coulombs}]}\frac{[\text{Coulombs}]}{[\text{Seconds}][\text{Length}]}=[\text{Force}]\frac{[\text{Length}]}{[\text{Seconds}][\text{Length}]^2}=\frac{[\text{Power}]}{[\text{Area}]}$$

Note: this is almost an intensity which is given by: $I=<\vec{S}>\cdot\hat{n}$ .

Where $u_{EM}=\frac{1}{2\mu_0}\vec{B}\cdot\vec{B}+\frac{1}{2}\epsilon_0\vec{E}\cdot\vec{E}:U_{EM}=\oiiint u_{EM}d^3r$ $u_{EM}$ is the EM energy density.

$$\frac{dW}{dt}=\oiiint\left[-\frac{1}{2\mu_0}\frac{\partial B^2}{\partial t}-\frac{\epsilon_0}{2}\frac{\partial E^2}{\partial t}-\vec{\nabla}\cdot\vec{S}\right]d^3r$$

Now consider a closed system in which no external work is done. Then:

$$\frac{\partial u_{EM}}{\partial t}=-\vec{\nabla}\cdot\vec{S}$$

This says then the time rate of energy change is equal to the negative divergence of the Poynting vector. This is it in differential form. In integral form:

$$\frac{\partial}{\partial t}\oiiint U_{EM}d^3r=\oiiint\vec{\nabla}\cdot\vec{S}d^3r\Rightarrow\frac{\partial U_{EM}}{\partial t}=-\oiint\vec{S}\cdot d\vec{A}$$

This in words is the time rate of change of the EM energy is equal to the flux of the Poynting vector through a closed surface containing the energy density that is changing. The Poynting vector is the flux of energy per unit time: J/(s $m^2$).

## Example 8.1

Your author considers a wire held at a potential difference V of length L and radius a. A current flows in the wire (in the z direction). The electric field in the wire is given by:

$$\vec{E}=\frac{V}{L}\hat{z}$$

By Ampere's law, at the surface of the wire, the magnetic field is given by:

$$\vec{B}=\frac{\mu_0 I}{2\pi a}\hat{\phi}$$

At the surface of the wire, the Poynting vector is given by:

$$\vec{S}=\frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a}\left(\hat{z}\,x\,\hat{\phi}\right)$$

The cross product is :

$$\hat{z}\,x\,\hat{\phi}=\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ 0 & 0 & 1\\ -\sin\phi & \cos\phi & 0\end{bmatrix}=\hat{x}(-\cos\phi)-\hat{y}(\sin\phi)=-\hat{s}$$

$$\vec{S}=-\frac{1}{\mu_0}\frac{V}{L}\frac{\mu_0 I}{2\pi a}\hat{s}$$

Integrate over the surface of the wire:

$$\oiint \vec{S}\cdot d\vec{A}=\frac{--IV}{2\pi a L}\oiint a\,d\phi\,dz=IV$$

(the normal to the area points outward, so that the negative cancels.

But the power radiated is given by $P=\frac{\partial U_{EM}}{\partial t}=IV$

Which is pretty much exactly what we obtained with the power radiated, for example by a resistor. But you'll note here we did not need to refer specifically to something like the wire had a resistance; although the deeper truth is that we did. Where this came in was in fact in assumption of the (finite) current I that exists:

$$\vec{J}=\sigma\vec{E}\Rightarrow\oiint\vec{J}\cdot\vec{A}=\sigma\oiint\vec{E}\cdot d\vec{A}=I\Rightarrow I=\sigma\frac{V}{L}(2\pi a)=\frac{1}{\rho}\frac{V}{L}(2\pi a)\Rightarrow I\left[\rho\frac{L}{A}\right]=V\Rightarrow V=IR$$

This permits writing the power in the three forms:

$$P=IV=I^2R=\frac{V^2}{R}$$