

7.2.3 Inductance

consider two loops of wire which are spatially separated but at rest with respect to each other. In the first look, supply a current I_1 which will result in a magnetic field from the first loop B_1 . We calculate the magnetic field from Biot-Savart:

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{L}_1 \times \hat{r}_{ip}}{r_{ip}^2}$$

The flux passing through the second loop is:

$$\Phi_2 = \iint \vec{B}_1 \cdot d\vec{a}_2$$

The flux passing through loop 2 is proportional to the current in loop 1. We define the mutual inductance of the two loops based upon this:

$$\Phi_2 = M_{21} I_1$$

where M_{21} is called the "mutual inductance" of the loops.

There is a standard way that the mutual inductance can be expressed:

$$\Phi_2 = \iint \vec{B}_1 \cdot d\vec{a}_2 = \iint (\vec{\nabla} \times \vec{A}_1) \cdot d\vec{a}_2 = \oint \vec{A}_1 \cdot d\vec{L}_2$$

But the vector potential is given by:

$$\vec{A}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{L}_1}{r_{ip}}$$

So we have:

$$\Phi_2 = \frac{\mu_0}{4\pi} I_1 \oint \left(\oint \frac{d\vec{L}_1}{r_{ip}} \right) \cdot d\vec{L}_2$$

So we have the mutual inductance:

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{L}_1 \cdot d\vec{L}_2}{r_{ip}}$$

Your author refers to this as the Neumann formula and from this it is clear that the mutual inductance depends upon the shapes and positions of the two loops and $M_{21} = M_{12}$.

Example 7.10 A short solenoid with radius a and n_1 turns per unit length is inside a much longer solenoid with n_2 turns per unit length. If the inner one carries a current I_1 , what is the mutual inductance of the system?

If you do this problem from this viewpoint, it is hard. Instead we apply a current I_2 to the long solenoid and calculate the mutual inductance that way.

The magnetic field inside the second solenoid is given by: $B = \mu_0 n_2 I_2$ so the flux through the short solenoid is given by:

$$\Phi_M = (n_1 h_1) (\mu_0 n_2 I_2) (\pi a^2) \Rightarrow M_{12} = M_{21} = \mu_0 \pi a^2 n_1 n_2 h$$

If we now allow the current to vary by Faraday's law we then have:

$$\text{emf}_2 = -\frac{d\Phi_2}{dt} = -M \frac{dI_1}{dt}$$

Now this changing current also introduces a emf in the circuit itself by:

$$\Phi = LI \Rightarrow \text{emf} = -L \frac{dI}{dt}$$

where L is the "self inductance".

Example 7.11: find the self-inductance of a toroid with a rectangular cross section (inner radius a , outer radius b , height h , N turns).

From Ampere's law, the magnetic field in the toroid is:

$$B = \frac{\mu_0 NI}{2\pi s}$$

The total flux through the toroid is then:

$$\Phi_M = N \oint \vec{B} \cdot d\vec{a} = \frac{\mu_0 N^2 I}{2\pi} h \int_a^b \frac{ds}{s} = \frac{\mu_0 N^2 I h}{2\pi} \ln\left(\frac{b}{a}\right) \Rightarrow L = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

7.2.4 Energy in magnetic fields

We start by calculating the work required to establish a current in an inductance:

$$\text{Power} = \frac{dU_M}{dt} = -I\mathcal{E} = LI \frac{dI}{dt} \Rightarrow U_M = \frac{1}{2} LI^2$$

This is the stored magnetostatic energy when a current I is established.

We can obtain the magnetostatic energy density by this:

$$\Phi = \oint \vec{B} \cdot d\vec{a} = \oint (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint \vec{A} \cdot d\vec{I} \Rightarrow LI = \oint \vec{A} \cdot d\vec{I}$$

The magnetostatic energy is then:

$$U_M = \frac{1}{2} I \oint \vec{A} \cdot d\vec{I} = \frac{1}{2} \oint (\vec{A} \cdot \vec{I}) dI = \frac{1}{2} \iiint (\vec{A} \cdot \vec{J}) d^3r = \frac{1}{2\mu_0} \iiint \vec{A} \cdot (\vec{\nabla} \times \vec{B}) d^3r$$

$$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \Rightarrow \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

Now integration gives:

$$U_M = \frac{1}{2\mu_0} \left[\iiint B^2 d^3r - \oint (\vec{A} \times \vec{B}) \cdot d\vec{a} \right]$$

$$\text{because } \iiint \vec{\nabla} \cdot (\vec{A} \times \vec{B}) d^3r = \oint (\vec{A} \times \vec{B}) \cdot d\vec{a}$$

What is now normally argued is to let the surface go to infinity where B will vanish so that the magnetostatic energy is expressed as:

$$U_M = \frac{1}{2\mu_0} \iiint_{\text{all space}} B^2 d^3r \quad \text{which is similar in concept to} \quad U_E = \frac{1}{2} \epsilon_0 \iiint E^2 d^3r .$$

We thus identify the magnetostatic energy density as:

$$u_M = \frac{B^2}{2\mu_0} \quad \text{which is again similar in concept to the electrostatic form:} \quad u_E = \frac{1}{2} \epsilon_0 E^2 .$$

Example 7.13 Long coax cable (inner radius a , outer radius b , length h) carries I inside and $-I$ outside. Find the magnetostatic energy.

$$B = \frac{\mu_0 I}{2\pi s} \Rightarrow u_M = \frac{\left(\frac{\mu_0 I}{2\pi}\right)^2}{2\mu_0} (2\pi h) \int_{s=a}^{s=b} \frac{s ds}{s^2} = \frac{(\mu_0 I^2)}{4\pi} h \ln\left(\frac{b}{a}\right)$$

Your author points out, and correctly so, that we could now calculate the inductance of the cable by:

$$U = \frac{1}{2} LI^2 \Rightarrow L = \frac{\mu_0 h}{2\pi} \ln\left(\frac{b}{a}\right) .$$