derivation of the continuity equation 2016

Consider a current density $\vec{J} \equiv \frac{d\vec{I}}{dA_{\perp}}$

You can imagine a wire of radius a carrying a current in the z direction. The perpendicular area is the cross section of the wire.

If you wish this to be connected to charge density in motion, it is: $\vec{J} = \rho\vec{v}$.

We can find the total current by integrating the current density:

$$\vec{I} = \oiint \vec{J} \cdot d\vec{A}$$

but we can now use Gauss's theorem to write this as a divergence:

$$\oiint \vec{J} \cdot d\vec{A} = \oiiint \left(\vec{\nabla} \cdot \vec{J}\right) d\tau$$

In magnetostatics, the currents arise only from charges in motion and displacement currents are not present. Then what ever charges leave an area come at the expense of the charges inside the volume of that area. And, remember we are calculating total current here. So:

$$\oiiint \left(\vec{\nabla} \cdot \vec{J}\right) d\tau = -\frac{\partial}{\partial t} \oiiint \rho \, d\tau \Rightarrow \vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

This is the continuity equation. Note, however, that it assumes that the only source of currents is from charges. In fact, displacement currents also exist and these currents arise from changing electric flux so you will expect that the continuity equation will not be complete in all circumstances. In static situations, it ought to be just fine.

You might want to know how to correctly write different current distributions:

$$\sum_{i=1}^{N} () q_i \vec{v}_i : \oint_{line} () \vec{I} \, dl : \oiint_{surface} () \vec{K} \, dA : \oiiint_{volume} () \vec{J} \, d\tau$$

These provide you with the translations needed in different situations.

Finally, you will want to note problem 5.7 which shows how currents arise from a decrease in dipole moment in some region of space. In particular:

$\oiiint \vec{J} \, d\tau = -\left(\frac{\partial \vec{p}}{\partial t}\right)$ where $\vec{p}$ is the total dipole moment of a sample. These depolarization currents are quite real and measurable. However since the source of a dipole is associated with charges, the continuity equation does apply.