Multipole Expansions

The potential from an electric dipole is given by:

$$V\left(\vec{r}_p\right) = kq\left(\frac{1}{r_{+p}} - \frac{1}{r_{-p}}\right)$$

We can re-repress this:

We'll let the dipole be oriented along the z axis with a separation of d and the positive end of the dipole is at +d/2.

$$\vec{r}_+ = \frac{d}{2}\hat{z} \Rightarrow \vec{r}_{+p} = x_p\hat{x} + y_p\hat{y} + \left(z_p - \frac{d}{2}\right)\hat{z}$$

$$\vec{r}_- = -\frac{d}{2}\hat{z} \Rightarrow \vec{r}_{-p} = x_p\hat{x} + y_p\hat{y} + \left(z_p + \frac{d}{2}\right)\hat{z}$$

Thus:

$$\left|\vec{r}_{+p}\right| = \sqrt{x_p^2 + y_p^2 + z_p^2 - z_p d + \frac{d^2}{4}} = \sqrt{r_p^2 + \frac{d^2}{4} - 2\vec{r}_p \bullet \vec{r}_+}$$

$$\left|\vec{r}_{-p}\right| = \sqrt{x_p^2 + y_p^2 + z_p^2 + z_p d + \frac{d^2}{4}} = \sqrt{r_p^2 + \frac{d^2}{4} + 2\vec{r}_p \bullet \vec{r}_-}$$

We are going to investigate the region rp>>d.

$$\frac{1}{\left|\vec{r}_{+p}\right|} = \frac{1}{\sqrt{r_p^2 + \frac{d^2}{4} - 2\vec{r}_p \bullet \vec{r}_+}} = \frac{1}{r_p}\frac{1}{\sqrt{1 + \left(\frac{d}{2r_p}\right)^2 - \frac{2\vec{r}_p \bullet \vec{r}_+}{r_p^2}}} \approx \frac{1}{r_p}\left(1 + \left(\frac{d}{2r_p}\right)^2 + \frac{\vec{r}_p \bullet \vec{r}_+}{r_p^2}\right)$$

$$\left|\vec{r}_{-p}\right| = \frac{1}{\sqrt{r_p^2 + \frac{d^2}{4} + 2\vec{r}_p \bullet \vec{r}_-}} = \frac{1}{r_p}\frac{1}{\sqrt{1 + \left(\frac{d}{2r_p}\right)^2 + \frac{2\vec{r}_p \bullet \vec{r}_-}{r_p^2}}} = \frac{1}{r_p}\left(1 + \left(\frac{d}{2r_p}\right)^2 - \frac{\vec{r}_p \bullet \vec{r}_-}{r_p^2}\right)$$

We can subtract these to obtain:

$$\frac{1}{r_{+p}} - \frac{1}{r_{-p}} = \frac{1}{r_p}\left(\frac{\vec{r}_p \bullet \vec{r}_+}{r_p^2} - \frac{\vec{r}_p \bullet \vec{r}_-}{r_p^2}\right)$$

When written in this form, it's pretty obvious that this is correct up to much higher terms. It is not necessary to ignore the d/2rp term so early.

$$\vec{r}_p \bullet \vec{r}_+ = r_p \frac{d}{2}\cos(\theta)$$

$$\vec{r}_p \bullet \vec{r}_- = -r_p \frac{d}{2}\cos(\theta)$$

We thus have:

$$\frac{1}{r_{+p}} - \frac{1}{r_{-p}} = \frac{1}{r_p}\left(\frac{\vec{r}_p \bullet \vec{r}_+}{r_p^2} - \frac{\vec{r}_p \bullet \vec{r}_-}{r_p^2}\right) = \frac{d\cos(\theta)}{r_p^2}$$

This gives us the dipole potential:

$$V\left(\vec{r}_p\right) = kq\frac{d\cos(\theta)}{r_p^2}$$

What we want to do is to develop a systematic method for the expansion of the potential for any number of charge pairs placed together. (look at figure 3.27 and caption)

For a more general distribution of charge, we have:

$$V\left(\vec{r}_p\right) = k\int_{\substack{\text{all} \\ \text{charges}}} \frac{\rho\left(\vec{r}_i\right)}{\left|\vec{r}_{ip}\right|}d\tau_i$$

we express the distance as:

$$\left(\vec{r}_{ip}\right)^2 = \left(\vec{r}_p - \vec{r}_i\right)\bullet\left(\vec{r}_p - \vec{r}_i\right) = \vec{r}_p \bullet \vec{r}_p + \vec{r}_i \bullet \vec{r}_i - 2\vec{r}_p \bullet \vec{r}_i = r_p^2 + r_i^2 - 2\left|\vec{r}_p\right|\left|\vec{r}_i\right|\cos\left(\theta_{ip}\right)$$

Thus,

$$\left(\vec{r}_{ip}\right)^2 = r_p^2\left[1+\left(\frac{r_i}{r_p}\right)^2 - 2\left(\frac{r_i}{r_p}\right)\cos\left(\theta_{ip}\right)\right]$$

where the angle is between the charge vector and the point vector.

Following your author: let

$$\varepsilon \equiv \left(\frac{r_i}{r_p}\right)\left(\frac{r_i}{r_p} - 2\cos\left(\theta_{ip}\right)\right)$$

Then

$$\left|\vec{r}_{ip}\right| = \left|\vec{r}_p\right|\sqrt{1+\varepsilon}$$

We then express the inverse distance as:

$$\frac{1}{\left|\vec{r}_{ip}\right|} = \frac{1}{\left|\vec{r}_p\right|}\left(1+\varepsilon\right)^{-\frac{1}{2}} \approx \frac{1}{\left|\vec{r}_p\right|}\left[1 - \frac{\varepsilon}{2} + \frac{3\varepsilon^2}{8} - \frac{5\varepsilon^3}{16} + \ldots\right]$$

We could, of course, write this as:

$$\frac{1}{\left|\vec{r}_{ip}\right|} \approx \frac{1}{\left|\vec{r}_p\right|}\left[1 - \frac{1}{2}\left(\frac{r_i}{r_p}\right)\left(\frac{r_i}{r_p} - 2\cos\left(\theta_{ip}\right)\right) + \frac{3}{8}\left[\left(\frac{r_i}{r_p}\right)\left(\frac{r_i}{r_p} - 2\cos\left(\theta_{ip}\right)\right)\right]^2 - \frac{5}{16}\left[\left(\frac{r_i}{r_p}\right)\left(\frac{r_i}{r_p} - 2\cos\left(\theta_{ip}\right)\right)\right]^3 + \ldots\right]$$

We'd want to write this in powers of ri/rp as:

$$\frac{1}{\left|\vec{r}_{ip}\right|} \approx \frac{1}{\left|\vec{r}_p\right|}\left[\begin{array}{l} 1+\left(\frac{r_i}{r_p}\right)\cos\left(\theta_{ip}\right) - \frac{1}{2}\left(\frac{r_i}{r_p}\right)^2 \\ \frac{3}{8}\left[\left(\frac{r_i}{r_p}\right)^2\left(\left(\frac{r_i}{r_p}\right)^2 - 4\frac{r_i}{r_p}\cos\left(\theta_{ip}\right) + 4\cos^2\left(\theta_{ip}\right)\right)\right] \\ -\frac{5}{16}\left(\frac{r_i}{r_p}\right)^4\left[\left(\frac{r_i}{r_p}\right)^2 - 4\frac{r_i}{r_p}\cos\left(\theta_{ip}\right) + 4\cos^2\left(\theta_{ip}\right)\right] \\ +\frac{5}{16}\left(\frac{r_i}{r_p}\right)^3\left[2\left(\frac{r_i}{r_p}\right)^2\cos\left(\theta_{ip}\right) - 8\frac{r_i}{r_p}\cos^2\left(\theta_{ip}\right) + 8\cos^3\left(\theta_{ip}\right)\right] \end{array}\right]$$

$$\frac{1}{\left|\vec{r}_p\right|} \approx \frac{1}{\left|\vec{r}_p\right|}\left[\begin{array}{l} 1 \\ +\left(\frac{r_i}{r_p}\right)\cos\left(\theta_{ip}\right) \\ -\frac{1}{2}\left(\frac{r_i}{r_p}\right)^2 + \frac{3}{2}\left(\frac{r_i}{r_p}\right)^2\cos^2\left(\theta_{ip}\right) \\ -\frac{3}{2}\left(\frac{r_i}{r_p}\right)^3\cos\left(\theta_{ip}\right) + \frac{10}{2}\left(\frac{r_i}{r_p}\right)^3\cos^3\left(\theta_{ip}\right) \\ +\ldots \end{array}\right]$$

$$\frac{1}{\left|\vec{r}_p\right|} \approx \frac{1}{\left|\vec{r}_p\right|}\left[\begin{array}{l} 1+\left(\frac{r_i}{r_p}\right)\cos\left(\theta_{ip}\right) + \left(\frac{r_i}{r_p}\right)^2\left(3\cos^2\left(\theta_{ip}\right) - 1\right)/2 + \\ \left(\frac{r_i}{r_p}\right)^3\left(5\cos^3\left(\theta_{ip}\right) - 3\cos\left(\theta_{ip}\right)\right)/2 + \ldots \end{array}\right]$$

Interestingly enough, we can thus group the series into a series of Legendre Polynomials as:

$$\frac{1}{\left|\vec{r}_{ip}\right|} \approx \frac{1}{\left|\vec{r}_p\right|}\sum_{n=0}^{\infty}\left(\frac{\left|\vec{r}_i\right|}{\left|\vec{r}_p\right|}\right)^n P_n\left(\cos\theta_{ip}\right)$$

We use this in the general form of the potential:

$$V(\vec{r}_p) = k \int_{\substack{\text{all}\\\text{charges}}} \rho(\vec{r}_i) \frac{1}{|\vec{r}_p|} \sum_{n=0}^{\infty} \left(\frac{|\vec{r}_i|}{|\vec{r}_p|} \right)^n P_n(\cos\theta_{ip}) d\tau_i = k \sum_{n=0}^{\infty} \frac{1}{|\vec{r}_p|^{n+1}} \int_{\substack{\text{all}\\\text{charges}}} |\vec{r}_i|^n P_n(\cos\theta_{ip}) \rho(\vec{r}_i) d\tau_i$$

As written, this is pretty much an exact solution to the potential of an arbitrary charge distribution … but remember, we're specifying something about azimuthal symmetry in this presentation. In other words, if something depends upon φ, you're going to have a more complicated situation on your hands. Sometimes, you might be able to superimpose monopole potentials onto the result to build up asymmetric distributions.

This is referred to as the multipole expansion of the potential.
Incidentally, we could get this result another way:
A generating function for the Legendre polynomials is given on our favorite site:
http://mathworld.wolfram.com/LegendrePolynomial.html

$$g(t, x) = (1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(x) t^n. \qquad \text{(eq. 11)}$$

We write this as:

$$\frac{1}{|\vec{r}_p| \sqrt{1 + \left(\frac{|\vec{r}_i|}{|\vec{r}_p|} \right)^2 - 2 \left(\frac{|\vec{r}_i|}{|\vec{r}_p|} \right) \cos(\theta_{ip})}} = \frac{1}{|\vec{r}_p|} \sum_{n=0}^{\infty} \left(\frac{|\vec{r}_i|}{|\vec{r}_p|} \right)^n P_n(\cos(\theta_{ip}))$$

Thus the potential becomes:

$$V(\vec{r}_p) = k \sum_{n=0}^{\infty} \frac{1}{|\vec{r}_p|^{n+1}} \int_{\substack{\text{all}\\\text{charges}}} |\vec{r}_i|^n P_n(\cos\theta_{ip}) \rho(\vec{r}_i) d\tau_i$$

For me, this is much more pleasing since the other method did not really derive anything. We still have the symmetry requirement, however!

Let me now work a problem for you.
Problem 3.26; A sphere of radius R, centered at the origin, carries charge density:

$$\rho(r_i, \theta_i) = c \frac{R}{r_i^2} (R - 2r_i) \sin(\theta_i)$$

where c is a constant. Find the approximate potential for points on the z-axis far away from the sphere.

Solution:

$$V(\vec{r}_p) = kcR \sum_{n=0}^{\infty} \frac{1}{|\vec{r}_p|^{n+1}} \int_{\substack{\text{all}\\\text{charges}}} r_i^{n-2} P_n(\cos\theta_{ip}) (R - 2r_i) \sin(\theta_i) d\tau_i$$

Thus, we have angular and radial integrals:

$$V(\vec{r}_p) = 2\pi kcR \sum_{n=0}^{\infty} \frac{1}{|\vec{r}_p|^{n+1}} \left[\int_{r_i=0}^{r_i=R} r_i^n (R - 2r_i) dr_i \int_{\theta=0}^{\pi} P_n(\cos\theta_{ip}) \sin^2(\theta_i) d\theta_i \right]$$

Along the +z-axis, $\theta_{ip} = \theta_i$

This last statement is one subtle puppy … if it's not true that the angle between p and the charge is the same as the polar angle, things are going to get harder. I recommend that you try to avoid having this not true if at all possible by coordinate transformations!

The radial integral is pretty easy to evaluate in general:

$$\int_{r_i=0}^{r_i=R} r_i^n\left(R-2r_i\right)dr_i=\left[\frac{Rr_i^{n+1}}{n+1}-2\frac{r_i^{n+2}}{n+2}\right]\Bigg|_0^R=R^{n+2}\left[\frac{1}{n+1}-\frac{2}{n+2}\right]=R^{n+2}\left[\frac{n+2-2n-2}{(n+1)(n+2)}\right]=\frac{-nR^{n+2}}{(n+1)(n+2)}$$

Our potential then looks like this:

$$V\left(\vec{r}_p\right)=\sum_{n=0}^{\infty}\left[\frac{-2\pi kcnR^{n+3}}{r_p^{n+1}(n+1)(n+2)}\right]\left[\int_{\theta=0}^{\pi}P_n\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i\right]$$

Let's look now at the angular integral:

$$I\equiv\int_{\theta=0}^{\pi}P_n\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i$$

There might be lots of ways to approach this problem at this point. I suggest that you do a term-by-term evaluation of this integral.

$$I\equiv\int_{\theta=0}^{\pi}P_n\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i$$

Here is the intent of this problem .. we're going to have a bit of trouble evaluating this in general. We can, however, do it on a term-by-term basis, which is what I'm going to do.

Let's evaluate the radial integral first.

$$\int_{r_i=0}^{r_i=R} r_i^n\left(R-2r_i\right)dr_i=R\left.\frac{r_i^{n+1}}{n+1}\right|_0^R-2\left.\frac{r_i^{n+2}}{n+2}\right|_0^R=\frac{R^{n+2}}{n+1}-2\frac{R^{n+2}}{n+2}=$$

$$=R^{n+2}\left[\frac{1}{n+1}-\frac{2}{n+2}\right]=R^{n+2}\left[\frac{n+2-2n-2}{(n+1)(n+2)}\right]=\left[\frac{-nR^{n+2}}{(n+1)(n+2)}\right]$$

Our integral then looks like this:

$$V\left(\vec{r}_p\right)=\sum_{n=0}^{\infty}\left[\frac{-2\pi kcnR^{n+3}}{r_p^{n+1}(n+1)(n+2)}\right]\left[\int_{\theta=0}^{\pi}P_n\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i\right]$$

now, let's evaluate the angular integral.

$$I_n\equiv\int_{\theta=0}^{\pi}P_n\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i$$

$$I_0\equiv\int_{\theta=0}^{\pi}P_0\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i=\int_{\theta=0}^{\pi}\sin^2\left(\theta_i\right)d\theta_i=\left[\frac{\theta}{2}-\frac{\sin(2\theta)}{4}\right]\Bigg|_0^{\pi}=\frac{\pi}{2}$$

$$I_1\equiv\int_{\theta=0}^{\pi}P_1\left(\cos\theta_i\right)\sin^2\left(\theta_i\right)d\theta_i=\int_{\theta=0}^{\pi}\cos(\theta)\sin^2\left(\theta_i\right)d\theta_i=\left[\frac{\sin(\theta)}{4}-\frac{\sin(3\theta)}{12}\right]\Bigg|_0^{\pi}=0$$

$$I_2 \equiv \int_{\theta=0}^{\pi} P_2(\cos\theta_i)\sin^2(\theta_i)d\theta_i =$$

$$\int_{\theta=0}^{\pi} \tfrac{3}{2}\cos^2(\theta)\sin^2(\theta_i)d\theta_i - \int_{\theta=0}^{\pi} \tfrac{1}{2}\sin^2(\theta_i)d\theta_i = \tfrac{3}{2}\left[\tfrac{\theta}{8} - \tfrac{\sin(4\theta)}{32}\right]\Big|_0^{\pi} - \tfrac{\pi}{4} = -\tfrac{1}{16}\pi$$

$$I_3 \equiv \int_{\theta=0}^{\pi} P_3(\cos\theta_i)\sin^2(\theta_i)d\theta_i = \int_{\theta=0}^{\pi} \tfrac{5}{2}\cos^3(\theta_i)\sin^2(\theta_i)d\theta_i - \int_{\theta=0}^{\pi} \tfrac{3}{2}\cos(\theta_i)\sin^2(\theta_i)d\theta_i =$$

$$= \tfrac{5}{2}\left[\tfrac{\sin(\theta)}{8} - \tfrac{\sin(3\theta)}{48} - \tfrac{\sin(5\theta)}{80}\right]\Big|_0^{\pi} - \tfrac{3}{2}\left[\tfrac{\sin(\theta)}{4} - \tfrac{\sin(3\theta)}{12}\right]\Big|_0^{\pi} = 0$$

$$I_4 \equiv \int_{\theta=0}^{\pi} P_4(\cos\theta_i)\sin^2(\theta_i)d\theta_i =$$

$$\int_{\theta=0}^{\pi} \tfrac{35}{8}\cos^4(\theta_i)\sin^2(\theta_i)d\theta_i - \int_{\theta=0}^{\pi} \tfrac{30}{8}\cos^2(\theta_i)\sin^2(\theta_i)d\theta_i + \int_{\theta=0}^{\pi} \tfrac{3}{8}\sin^2(\theta_i)d\theta_i$$

$$= \tfrac{35}{8}\tfrac{\pi}{16} - \tfrac{30}{8}\tfrac{\pi}{8} + \tfrac{\pi}{2} = \pi\left(\tfrac{35}{128} - \tfrac{60}{128} + \tfrac{64}{128}\right) = \tfrac{39}{128}\pi$$

$$I_5 \equiv \int_{\theta=0}^{\pi} P_5(\cos\theta_i)\sin^2(\theta_i)d\theta_i =$$

$$\int_{\theta=0}^{\pi} \tfrac{63}{8}\cos^5(\theta_i)\sin^2(\theta_i)d\theta_i - \int_{\theta=0}^{\pi} \tfrac{70}{8}\cos^3(\theta_i)\sin^2(\theta_i)d\theta_i + \int_{\theta=0}^{\pi} \tfrac{15}{8}\cos(\theta_i)\sin^2(\theta_i)d\theta_i =$$

$$= 0 - 0 + 0 = 0$$

$$I_6 \equiv \int_{\theta=0}^{\pi} P_6(\cos\theta_i)\sin^2(\theta_i)d\theta_i =$$

$$\int_{\theta=0}^{\pi} \tfrac{231}{16}\cos^6(\theta_i)\sin^2(\theta_i)d\theta_i - \int_{\theta=0}^{\pi} \tfrac{315}{16}\cos^4(\theta_i)\sin^2(\theta_i)d\theta_i + \int_{\theta=0}^{\pi} \tfrac{105}{16}\cos^2(\theta_i)\sin^2(\theta_i)d\theta_i - \int_{\theta=0}^{\pi} \tfrac{5}{16}\sin^2(\theta_i)d\theta_i =$$

$$= \tfrac{231}{16}\tfrac{5\pi}{128} - \tfrac{315}{16}\tfrac{\pi}{16} + \tfrac{105}{16}\tfrac{\pi}{8} - \tfrac{\pi}{2} = \pi\left(\tfrac{1155}{2048} - \tfrac{315}{256} + \tfrac{105}{128} - \tfrac{1}{2}\right) = \pi\left(\tfrac{1155-2520+1680-1024}{2048}\right) = -\tfrac{709}{2048}\pi$$

You can reasonably assume I7=0.

Thus we have the solution:

$$V(\vec{r}_p) = 2\pi kc\left[0:0 - 1:\tfrac{R^4}{6r_p^2}0 + 2:\tfrac{2R^5}{15r_p^3}\tfrac{\pi}{16} + 3:0 + 4:\tfrac{8R^7}{30r_p^5}\tfrac{39\pi}{128} + 5:0 - 6:\tfrac{6R^9}{56r_p^7}\tfrac{709\pi}{2048} + 7:0\right]$$

Thus, to 1/r^8, we have:

$$V(\vec{r}_p) = 2\pi^2 kc\left[\tfrac{R^5}{120r_p^3} + \tfrac{312R^7}{3840r_p^5} - \tfrac{4254R^9}{114688r_p^7}\right]$$

And that’s how you do these type of problems along the z-axis.

Let’s look at the various terms in the expansion.

$$V\left(\vec{r}_p\right) = k\sum_{n=0}^{\infty}\frac{1}{\left|\vec{r}_p\right|^{n+1}}\int\limits_{\substack{\text{all}\\\text{charges}}}\left|\vec{r}_i\right|^n P_n\left(\cos\theta_{ip}\right)\rho\left(\vec{r}_i\right)d\tau_i$$

(0) The monopole term.

At large r, the potential is dominated by the monopole term. Take the first term in the series expansion:

(n=0):

$$V_0\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|}\int\limits_{\substack{\text{all}\\\text{charges}}}\left|\vec{r}_i\right|^0 P_0\left(\cos\theta_{ip}\right)\rho\left(\vec{r}_i\right)d\tau_i$$

Your author dismisses the ip subscript a bit too easily, though I think.

In any event, for the monopole term, since the Legendre polynomial is 1, we don’t need to worry about this. Thus, the monopole term is:

$$V_0\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|}\int\limits_{\substack{\text{all}\\\text{charges}}}\left|\vec{r}_i\right|^0 P_0\left(\cos\theta_{ip}\right)\rho\left(\vec{r}_i\right)d\tau_i = \frac{kQ}{\left|\vec{r}_p\right|}$$

Let’s get the next term, the dipole term.

$$V_1\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|^2}\int\limits_{\substack{\text{all}\\\text{charges}}}\left|\vec{r}_i\right| P_1\left(\cos\theta_{ip}\right)\rho\left(\vec{r}_i\right)d\tau_i$$

Your author’s notation in describing the dipole potential really does fail at this point (page 149). He says:

θ’ is the angle between r’ and r. There is no disagreement from me except that in this notation, one may be misled into thinking that θ’ is the angle that the charge is from the z-axis. I prefer my notation to that of your author’s here. Since:

$$P_1\left(\cos\left(\theta_{ip}\right)\right) = \cos\left(\theta_{ip}\right)$$

we can rewrite V1 as:

$$V_1\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|^3}\int\limits_{\substack{\text{all}\\\text{charges}}}\rho\left(\vec{r}_i\right)\left(\vec{r}_i \bullet \vec{r}_p\right)d\tau_i$$

This in turn is expressed as:

$$V_1\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|^3}\vec{r}_p \bullet \int\limits_{\substack{\text{all}\\\text{charges}}}\vec{r}_i\rho\left(\vec{r}_i\right)d\tau_i$$

And we define the dipole moment of the distribution as:

$$\vec{p} \equiv \int\limits_{\substack{\text{all}\\\text{charges}}}\vec{r}_i\rho\left(\vec{r}_i\right)d\tau_i$$

The dipole potential then simplifies to:

$$V_{dip} \equiv V_1\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|^3}\vec{r}_p \bullet \vec{p}$$

The dipole moment for a continuous charge distribution is then pretty straight-forward to obtain. If you have discrete charges, then via the delta-function notation, you can see for a collection of m charges:

$$\vec{p} = \sum_{i=1}^{m} q_i \vec{r}_i$$

And, if you had only 2 charges (a +q and a –q) then:

$$\vec{p} = q\left(\vec{r}_- - \vec{r}_+\right) = q\vec{d}$$

where d is the vector pointing from the – charge toward the + charge.

We might as well face it …. no matter how you look at it, this definition of d is going to bother you. Let's see, however, if it's going to match up with the torque on a dipole:

$$\vec{\tau} = \vec{p} \times \vec{E}$$

Suppose you have an external electric field directed along the +z axis so that:

$$\vec{E} = E\hat{z}$$

Let's align our diple so that the + charge is at x=+d/2 and the minus charge is at –d/2. The electric field, remember, gives the net force per unit positive test charge. The + charge will tend to want to move along the +z direction while the – charge will tend to want to move along the –z direction. The net torque will be given by $\vec{\tau} = \sum \vec{r} \times \vec{F}$:

$$\vec{\tau} = \vec{\tau}_+ + \vec{\tau}_- = +q\tfrac{d}{2}E\left(\hat{x} \times \hat{z}\right) + -q\tfrac{d}{2}E\left(-\hat{x} \times \hat{z}\right)$$

we can easily calculate the cross products:

$$\hat{x} \times \hat{y} = \hat{z}$$

$$\hat{y} \times \hat{z} = \hat{x}$$

$$\hat{z} \times \hat{x} = \hat{y}$$

Thus, the vector torque is given by:

$$\vec{\tau} = \vec{\tau}_+ + \vec{\tau}_- = +q\tfrac{d}{2}E\left(-\hat{y}\right) + -q\tfrac{d}{2}E\left(\hat{y}\right) = -qdE\hat{y}$$

The change in angular momentum is in the direction of the applied torque. This would thus give a rotation with the +charge moving in the +z direction and the – charge moving in the –z direction (in this coordinate rotation, -y would be out of the screen while +y would be into the screen).

Now, if d points from the negative toward the positive charge, we see that this produces the same torque:

$$\vec{\tau} = q\vec{d} \times \vec{E} = qdE\left(\hat{x} \times \hat{z}\right) = -qdE\hat{y}$$

Thus the direction of d is in good agreement which commonly accepted conventions, although you need to realize that it is in the ***opposite direction to the electric field inside a dipole!***

Here is somewhat of an important distinction:

A **physical** dipole consists of equal and opposite charges separated by a distance d:

$$\vec{p} = q\vec{r}_{i+} - q\vec{r}_{i-} = q\left(\vec{r}_{i+} - \vec{r}_{i-}\right) = q\vec{d}$$

where d is the vector pointing from the negative charge towards the positive charge.

Yes? Remember this:

$$\vec{r}_{ip} = \vec{r}_p - \vec{r}_i$$

is the vector pointing from the ith charge towards the point p in space.

The question is this: if this definition for the physical dipole:

$$\vec{p} = \sum_{i=1}^{m} q_i \vec{r}_i$$

consistent with what we got for a physical dipole?

The answer:

we had:

$$V_{dip} \equiv V_1\left(\vec{r}_p\right) = \frac{k}{|\vec{r}_p|^3} \vec{r}_p \bullet \vec{p}$$

For this definition we then obtain:

$$V_{dip} = \frac{k}{|\vec{r}_p|^3} \vec{r}_p \bullet \sum_{i=1}^{2} q_i \vec{r}_i = \frac{qk\vec{r}_p \bullet \vec{d}}{|\vec{r}_p|^3} = \frac{qkd\cos(\theta)}{|\vec{r}_p|^2}$$

for the dipole pair.

In order to make this physical dipole into a "pure" dipole, you let d go to zero while q goes to infinity, keeping the product constant. In that limit, the approximate dipole potential becomes exact. Your author suggests that should you ever have doubts about the context of the word dipole, then use:

$$V_{dip} \equiv V_1\left(\vec{r}_p\right) = \frac{k}{|\vec{r}_p|^3} \vec{r}_p \bullet \vec{p}$$

3.4.3 origin of Coordinates in Multipole Expansions

If a point charge is located at the origin, it is a pure monopole.

Moving it from the origin, changes this result: it's no longer a pure monopole.

Suppose a charge is placed away from the origin at: $\vec{r}_i$. The exact potential from this charge is:

$$V\left(\vec{r}_p\right) = \frac{kq}{|\vec{r}_p - \vec{r}_i|}$$

which is not at all the same as the monopole potential:

$$V_0\left(\vec{r}_p\right) = \frac{k}{|\vec{r}_p|} \int_{\substack{\text{all} \\ \text{charges}}} |\vec{r}_i|^0 P_0\left(\cos\theta_{ip}\right) \rho\left(\vec{r}_i\right) d\tau_i = \frac{kq}{|\vec{r}_p|}$$

what would result with a charge density given by:

$$\rho\left(\vec{r}_i\right) = \frac{1}{4\pi} \delta\left(\vec{r}_i\right)$$

This charge when displaced will also have a dipole moment:

$$\vec{p} = \sum_{i=1}^{m} q_i \vec{r}_i = q\vec{r}_i$$

In particular, suppose the charge is located as in Figure 3.32 in your text. Then:

$$\vec{p} = qd\hat{y}$$

will be the dipole moment and the dipole term will look like:

$$V_{dip} \equiv V_1\left(\vec{r}_p\right) = \frac{k}{|\vec{r}_p|^3} \vec{r}_p \bullet qd\hat{y} = \frac{kqd}{|\vec{r}_p|^2} \sin\left(\theta\right) \sin\left(\varphi\right)$$

The monopole moment Q is not changed by this translation, however.

here is an important theorem:

In general the dipole moment will change if you shift the origin of coordinates. If, however, the total charge is zero, then the dipole moment is independent of the choice of the origin. (Said without proof). Can you prove this?

3.4.4 The electric field of a dipole

Let's assume that a dipole is located at the origin and points in the z direction. The dipole potential is then:

$$V_{dip} \equiv V_1\left(\vec{r}_p\right) = \frac{k}{\left|\vec{r}_p\right|^3}\vec{r}_p \bullet \vec{p}$$

The electric field is obtained by

$$\vec{E} = -\vec{\nabla}V$$

Our appropriate coordinate system here is spherical. Thus:

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin(\theta)}\frac{\partial T}{\partial \varphi}\hat{\varphi}$$

There is no azimuthal dependence here so:

$$\vec{\nabla}T = \frac{\partial T}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta}$$

We can also write the dipole as:

$$V_{dip} \equiv V_1\left(\vec{r}_p\right) = \frac{k}{r_p^2}p\cos(\theta)$$

Our derivative, of course, will be with respect to space (not charge) coordinates. Thus:

$$\vec{E} = E_r\hat{r} + E_\theta\hat{\theta} = \frac{2kp\cos(\theta)}{r^3}\hat{r} + \frac{kp\sin(\theta)}{r^3}\hat{\theta} = \frac{kp}{r^3}\left(2\cos(\theta)\hat{r} + \sin(\theta)\hat{\theta}\right)$$

Problem 3.33: Show that the electric field of a "pure" dipole can be written in the coordinate – free form:

$$\vec{E}_{dip}\left(\vec{r}_p\right) = \frac{k}{r_p^3}\left[3\left(\vec{p} \bullet \hat{r}_p\right)\hat{r}_p - \vec{p}\right]$$

Solution: the thing in brackets is what we've really got to worry about.

$$2p\cos(\theta)\hat{r} + p\sin(\theta)\hat{\theta} = 2\left(\vec{p} \bullet \hat{r}_p\right)\hat{r} = 3\left(\vec{p} \bullet \hat{r}_p\right)\hat{r} - p\cos(\theta)\hat{r} + p\sin(\theta)\hat{\theta}$$

But (looking at the useful page):

$$\vec{p} = p_z\hat{z} = p\cos(\theta)\hat{r} - \sin(\theta)\hat{\theta}$$

Thus:

$$2p\cos(\theta)\hat{r} + p\sin(\theta)\hat{\theta} = 2\left(\vec{p} \bullet \hat{r}_p\right)\hat{r}_p = 3\left(\vec{p} \bullet \hat{r}_p\right)\hat{r}_p - p\cos(\theta)\hat{r}_p + p\sin(\theta)\hat{\theta} = 3\left(\vec{p} \bullet \hat{r}_p\right)\hat{r}_p - \vec{p}$$