2.3.3-2.3.4 Poisson's Equation and Laplace's Equation

Since $\vec{E} = -\vec{\nabla} V$ and $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$, for those electric fields produced by static charges we have:

$$\vec{\nabla} \cdot (\vec{\nabla} V) \equiv \vec{\nabla} V^2 = -\frac{\rho}{\epsilon_0}$$

which is Poisson's equation. In charge-free regions of space and provided that any existing potentials in these regions of space arise from static distributions of charges, we have Laplace's equation which is:

$$\vec{\nabla}^2 V = 0$$

$$\vec{\nabla} \times (\vec{E}) = \vec{\nabla} \times (-\vec{\nabla} V) = 0$$

The curl of a gradient is always zero.

Also notice that since $\vec{E} = -\vec{\nabla} V$, we can add any constant (or function depending not upon the p coordinates) to the potential function and the result for E will be unchanged. This is particularly useful when dealing with distributions such as infinite wires and infinite planes of charge.

It is important to see how the potential for a charge distribution is constructed by looking a the potentials from point charges and then superimposing the potentials.

From the definition of the potential, we have:

$$V(\vec{r}_p) = -\int_{\infty}^{\vec{r}_p} \vec{E}_{ip} \cdot d\vec{l}_p = k \frac{q_i}{|\vec{r}_{ip}|}$$

now if there are several charges (designated by the i's), then

$V(\vec{r}_p) = k \sum_{i=1}^{i=n} \frac{q_i}{|\vec{r}_{ip}|}$ which for a continuous distribution becomes $V(\vec{r}_p) = k \int_{\text{all } q_i} \frac{dq_i}{|\vec{r}_{ip}|}$.

The important thing to note here is that the potential obtained from the definition comes by the integration over space coordinates p, but the potential obtained by integration over a charge distribution is obtained by integration over charge coordinate positions (along with the charge density function). Additionally when gradients and divergences are obtained, the derivatives are with respect to space coordinates and not charge coordinates.

We can thus find the potential of an arbitrary charge distribution by:

$$V(\vec{r}_p) = \int_{\text{all } q_i} k \frac{\rho(\vec{r}_i)}{|\vec{r}_{ip}|} d^3 r_i \rightarrow \int_{\text{all } q_i} k \frac{\sigma(\vec{r}_i)}{|\vec{r}_{ip}|} d^2 r_i \rightarrow \int_{\text{all } q_i} k \frac{\lambda(\vec{r}_i)}{|\vec{r}_{ip}|} d r_i \quad .$$

For reference, recall that the electric field is given by:

$$\vec{E}(\vec{r}_p) = \int_{\text{all } q_i} k \frac{\rho(\vec{r}_i)}{\vec{r}_{ip}^2} \hat{r}_{ip} d^3 r_i \rightarrow \int_{\text{all } q_i} k \frac{\sigma(\vec{r}_i)}{\vec{r}_{ip}^2} \hat{r}_{ip} d^2 r_i \rightarrow \int_{\text{all } q_i} k \frac{\lambda(\vec{r}_i)}{\vec{r}_{ip}^2} \hat{r}_{ip} d r_i \quad .$$

And,

$$\vec{E}_p = -\vec{\nabla}_p V(\vec{r}_p) : \vec{\nabla}_p \cdot \vec{E}_p = \frac{\rho(\vec{r}_i)}{\epsilon_0} : \vec{\nabla}_p^2 V(\vec{r}_p) = -\frac{\rho(\vec{r}_p)}{\epsilon_0} : \vec{\nabla}_p \times \vec{E}_p = \vec{0} \quad .$$