Find the potential inside and outside a spherical shell of charge

$$V\left(\vec{r}_p\right)=\int_{\substack{\text{all}\\\text{charges}}} k\frac{dq}{\sqrt{x^2+y^2+\left(z_p-z\right)^2}}$$

For a spherical shell of charge, the charge density is given by

$$dq=\sigma R^2\sin\left(\theta\right)d\theta d\varphi$$

The potential then becomes:

$$V\left(\vec{r}_p\right)=\int_{\substack{\text{all}\\\text{charges}}} k\frac{\sigma R^2\sin(\theta)d\theta d\varphi}{\sqrt{x^2+y^2+\left(z_p-z\right)^2}}$$

We need then to be able to re-express z since this depends upon the angle.

From the useful sheet:

$$x=r\sin\left(\theta\right)\cos\left(\varphi\right)$$

$$y=r\sin\left(\theta\right)\sin\left(\varphi\right)$$

$$z=r\cos\left(\theta\right)$$

thus:

$$x^2+y^2=R^2\sin^2\left(\theta\right)$$

$$V\left(\vec{r}_p\right)=\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi} k\frac{\sigma R^2\sin(\theta)d\theta d\varphi}{\sqrt{R^2\sin^2(\theta)+R^2\cos^2(\theta)-2Rz_p\cos(\theta)+z_p^2}}$$

we'll want to simplify this:

$$V\left(\vec{r}_p\right)=k\sigma R^2\int_{\varphi=0}^{\varphi=2\pi}\int_{\theta=0}^{\theta=\pi}\frac{\sin(\theta)d\theta d\varphi}{\sqrt{R^2-2Rz_p\cos(\theta)+z_p^2}}$$

The φ integration is particularly easy:

$$V\left(\vec{r}_p\right)=k2\pi\sigma R^2\int_{\theta=0}^{\theta=\pi}\frac{\sin(\theta)d\theta}{\sqrt{R^2-2Rz_p\cos(\theta)+z_p^2}}$$

$$V\left(\vec{r}_p\right)=k2\pi\sigma R^2\int_{\theta=0}^{\theta=\pi}\frac{\sin(\theta)d\theta}{\sqrt{R^2-2Rz_p\cos(\theta)+z_p^2}}$$

let:

$$x=2Rz_p\cos\left(\theta\right):dx=-2Rz_p\sin\left(\theta\right)d\theta:\theta=0\Rightarrow x=2Rz_p:x=\pi\Rightarrow x=-2Rz_p$$

$$V\left(\vec{r}_p\right)=k2\pi\sigma R^2\int_{x=2Rz_p}^{x=-2Rz_p}\frac{\frac{dx}{-2Rz_p}}{\sqrt{R^2+z_p^2-x}}=-\frac{k\pi\sigma R}{z_p}\int_{x=2Rz_p}^{x=-2Rz_p}\frac{dx}{\sqrt{R^2+z_p^2-x}}$$

at our favorite site, enter 1/((a-x)^(1/2))

the result: $-2\sqrt{a-x}$

Thus:

$$V\left(\vec{r}\right)=\frac{2\pi k\sigma R}{z_p}\sqrt{R^2+z_p^2-x}\Big|_{x=2Rz_p}^{x=-2Rz_p}$$

$$V\left(\vec{r}\right)=\frac{2\pi k\sigma R}{z_p}\sqrt{R^2+z_p^2+2Rz_p}-\frac{2\pi k\sigma R}{z_p}\sqrt{R^2+z_p^2-2Rz_p}$$

A little bit of algebra gives:

$$V\left(\vec{r}\right)=\frac{2\pi k\sigma R}{z_p}\left[\sqrt{\left(R+z_p\right)^2}-\sqrt{\left(R-z_p\right)^2}\right]$$

Case I: if $z_p>R$:

$$V\left(\vec{r}\right)=\frac{2\pi k\sigma R}{z_p}\left[R+z_p-\left(z_p-R\right)\right]=\frac{4\pi k\sigma R^2}{z_p}=k\frac{Q}{z_p}$$

Case II: if $z_p<R$:

$$V\left(\vec{r}\right)=\frac{2\pi k\sigma R}{z_p}\left[\left(R+z_p\right)-\left(R-z_p\right)\right]=\frac{2\pi k\sigma R}{z_p}\left[2z_p\right]=4\pi k\sigma R=k\frac{Q}{R}$$

Note that the two solutions join at $z_p=R$, as would be expected.

We can now easily find the electric field by:

$$\vec{E}=-\vec{\nabla}_{r_p}V$$

If we rewrite the result for case I, we clearly have:

$$V\left(\vec{r}_p\right)=k\frac{Q}{\left|\vec{r}_p\right|}$$

In spherical coordinates, we have from the useful sheet:

$$\vec{\nabla}V=\frac{\partial V}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta}+\frac{1}{r\sin(\theta)}\frac{\partial V}{\partial\varphi}\hat{\varphi}$$

Thus, in this case, we have:

$$\vec{E}=-\frac{\partial V}{\partial r_p}\hat{r}_p=-kQ\frac{\partial\left[\left(r_p^2\right)^{-\frac{1}{2}}\right]}{\partial r_p}\hat{r}_p=-kQ\left(-\frac{1}{2}\left(r_p^2\right)^{-\frac{3}{2}}\left(2r_p\right)\right)\hat{r}_p=\frac{kQ}{r_p^2}\hat{r}_p$$

I did the derivative in the unusual way in order to be sure to preserve any sign implications, which every now and then crop up.

Also notice that inside the sphere:

$$\vec{E}=0\hat{r}_p$$

We can get one more thing from this formulation. Recall:

$$\vec{\nabla}_{\vec{r}_p}\bullet\left(\varepsilon_0\vec{E}\right)=\rho\left(\vec{r}_i\right)$$

Let's apply this to our result to get an expression for the volume charge density.

$$\vec{\nabla}_{r_p} \bullet \vec{E} = kQ\vec{\nabla}_{r_p} \bullet \left(\frac{\hat{r}_{ip}}{\vec{r}_{ip}^{\,2}}\right)$$

But, we had:

$$\vec{\nabla}_{\vec{r}_p} \bullet \left(\frac{\hat{r}_{ip}}{\vec{r}_{ip}^{\,2}}\right) = 4\pi\delta^3\left(\vec{r}_i\right)$$

thus, we can write:

$$\rho\left(\vec{r}_i\right) = \varepsilon_0 kQ\left[4\pi\delta^3\left(\vec{r}_i\right)\right] = Q\delta^3\left(\vec{r}_i\right)$$

What this means is , after you think about it for a while, is that the potential and the electric field outside the sphere is exactly the same as would result from a point charge located at the origin. Hmmm … it is interesting to see that what we know must be the answer can clearly come from our work.

What about inside the sphere?

Inside the sphere, we have:

$$\vec{E} = 0\hat{r}_p \Rightarrow \vec{\nabla} \bullet \left(\varepsilon_0 \vec{E}\right) = 0$$

the meaning of which is that inside the sphere, the result acts like what you'd find from empty space. The big place that things change is at R (the radius of the sphere).

We thus have the following for the volume charge distribution:

$$\rho\left(\vec{r}_i\right) = Q\delta^3\left(\vec{r}_i\right) \ \left(\text{if } \left|\vec{r}_i\right| > R\right)$$

$$\rho\left(\vec{r}_i\right) = 0 \ \left(\text{if } \left|\vec{r}_i\right| < R\right)$$

I believe you can imagine combining these two results now to say:

$$\rho\left(\vec{r}_i\right) = Q\delta^3\left(\vec{r}_i - R\hat{r}_i\right)$$

What I have just proven to you is approximately the following statement:

in a charge free region of space, the charge distribution required to produce a particular electrostatic potential and a particular electrostatic field is not unique.

We're going to use this later in order to fix boundary conditions in a technique called the method of images.