Boundary Conditions and the Uniqueness Theorem

First Uniqueness Theorem: The solution to Laplace's equation in some volume V is uniquely determined if V is specified on the boundary surface S.

Proof:

Suppose in a bounded region there were 2 solutions to Laplace's equation:

$$\nabla^2 V_1 = 0 \text{ and } \nabla^2 V_2 = 0$$

both of which assume some particular value on the boundary. The difference is:

$$V_3 = V_1 - V_2 \Rightarrow \nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

At the boundary, the two solutions must be the same since they both have the same value. duu….

inside the boundary, however, since no local maxima or minima are permitted, we must also have that V_3 is zero inside the boundary. Thus, $V_1=V_2$.

This means that no matter how you came up with your solution to Laplace's equation, if it satisfies the boundary conditions, it's the solution you seek.

This can also be shown to be true for Poisson's equation.

This means:

If you know the charge density in a region of space and you know the potential on the boundary of that region of space, then V is uniquely determined.

Second Uniqueness Theorem

(I will refer you to your author's proof here for reading)

In a volume V surrounded by conductors and containing a specified charge density ρ, the electric field is uniquely determined if the total charge on each conductor is known.

Methods of solution

3.2 The method of images

I will easily bet that you've never seen this technique in all your experience so here it is. (this is well know to advanced physics students)

Here is the typical problem:

A point charge q is held a distance d above an infinite grounded conducting plane. What is the potential in the region above the plane.

At first glance, you probably would think about using something like Laplace's equation and setting the bc to be zero on the plane. This won't really tell you, however, about the field in the + region of space.

Ok, then you're going to try Poisson's equation for the point charge and try to make sure that the plane is grounded. This problem is simple enough so that you can probably do this. However, there is a most elegant solution to this problem.

Basically the method of images says that you can replace a given boundary condition by a distribution of charges that will give that boundary condition. We already know of one such system of charges that will give a plane which is of zero potential, namely the electric dipole. Let's place the positive charge at a distance z=d above the plane and the negative charge at a distance z=-d below the plane. Everywhere in the positive region of space, we can easily write down the solution to V:

$$V(\vec{r}_p) = kq\left[\frac{1}{\sqrt{x_p^2+y_p^2+(z_p-d)^2}} - \frac{1}{\sqrt{x_p^2+y_p^2+(z_p+d)^2}}\right]$$

Looking at this, what do we see in the region above the plane?
(1) V=0 on the plane and (2) we see a charge at a distance d above the plane.
Since that's exactly what the problem specified, this is the solution to the problem in the +z region of space.
Notice the importance of the uniqueness theorem in the last statement!

Now we can indeed find the induced surface charge on the conductor:
We defined the normal derivative as:

$$\frac{\partial V}{\partial n} = \vec{\nabla} V \bullet \hat{n}$$

and then the surface charge is given by:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}$$

Let's apply this to our solution:
The normal direction is the z direction, so the normal derivative of V is:

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial z}$$

Of course, we'll need to evaluate this at z=0 in order to get the induced charge.
Thus:

$$\frac{\partial V(\vec{r}_p)}{\partial z} = kq\left[-\frac{1}{2}\frac{2(z_p-d)}{\left(x_p^2+y_p^2+(z_p-d)^2\right)^{3/2}} - -\frac{1}{2}\frac{2(z_p+d)}{\left(x_p^2+y_p^2+(z_p+d)^2\right)^{3/2}}\right]$$

$$\left.\frac{\partial V(\vec{r}_p)}{\partial z}\right|_{z=0} = kq\left[\frac{d}{\left(x_p^2+y_p^2+d^2\right)^{3/2}} + \frac{d}{\left(x_p^2+y_p^2+d^2\right)^{3/2}}\right] = \frac{2kdq}{\left(x_p^2+y_p^2+d^2\right)^{3/2}}$$

In terms of the cylindrical coordinate s, we have:

$$\left.\frac{\partial V(\vec{r}_p)}{\partial z}\right|_{z=0} = \frac{2qkd}{\left(s^2+d^2\right)^{3/2}} = \frac{qd}{2\pi\varepsilon_0\left(s^2+d^2\right)^{3/2}}$$

The induced surface charge density is then:

$$\sigma = -\varepsilon_0 \frac{\partial V(\vec{r}_p)}{\partial z}\bigg|_{z=0} = \frac{-qd}{2\pi(s^2+d^2)^{3/2}}$$

One thing you ought to check is that the integral over the plane gives q.
Thus:

$$2\pi\int_{s=0}^{s=\infty} \sigma s ds = \frac{-qd}{2\pi(s^2+d^2)^{3/2}} = -qd\int_{s=0}^{s=\infty} \frac{sds}{(s^2+d^2)^{3/2}}$$

You can evaluate this at our favorite site:
enter: x/(x^2+d^2)^(3/2)

get: $-\frac{1}{\sqrt{d^2+x^2}}$

Thus:

$$2\pi\int_{s=0}^{s=\infty} \sigma s ds = \frac{-qd}{2\pi(s^2+d^2)^{3/2}} = \frac{qd}{\sqrt{x^2+d^2}}\bigg|_{x=0}^{x=\infty} = -\frac{qd}{d} = -q$$

The induced charge is thus equal to the charge that induced it and opposite in sign. You'll also note that the induced surface charge density is largest just under the inducing charge.

It's now very easy to calculate the force that the plane exerts on the point charge:

$$\vec{F} = -\frac{kq^2}{(2d)^2}\hat{z}$$

What is the total force that the original charge exerts on the plane?
Hmmm … remember Newton's third law?

Your author points out that you've got to be careful when you calculate the energy of the field in the +z region. There are a lot of pleasing arguments that we could use to get this energy. May I recommend that you'll probably want to do the direct integration here in order to obtain the correct result. Thus:

$$W = \int_{\infty}^{d} \vec{F}\bullet d\vec{l} = \int_{\infty}^{d} \frac{kq^2}{4z^2}dz = -\frac{kq^2}{4z}\bigg|_{z=\infty}^{z=d} = -\frac{kq^2}{4d}$$

(the work done on the induced charge is done for free since it's an equipotential surface).

We'll look at other image problems later.
Let me show you one of my favorite image problems.

Two infinite planes are separated by a distance 2d. A positive charge q is located between the two planes. Find the potential in the region between the two planes.

Solution:

locate your first real charge at x=0 (y=0,z=0).
Your first image is (-q,2d)&(-q,-2d)
You'll need to correct the effects of the images on the planes they're not supposed to interact with. The second image is (+q,-4d)&(+q,4d)
The third image is (-q,6d)&(-q,-6d)

etc.

You can see then that the potential is given by:

$$V\left(\vec{r}_p\right) = kq\left[\begin{array}{l} \frac{1}{\sqrt{x_p^2+y_p^2+z_p^2}} + \\ \frac{-1}{\sqrt{x_p^2+y_p^2+\left(z_p-2d\right)^2}} + \frac{-1}{\sqrt{x_p^2+y_p^2+\left(z_p+2d\right)^2}} \\ \frac{1}{\sqrt{x_p^2+y_p^2+\left(z_p-4d\right)^2}} + \frac{1}{\sqrt{x_p^2+y_p^2+\left(z_p+4d\right)^2}} \\ \frac{-1}{\sqrt{x_p^2+y_p^2+\left(z_p-6d\right)^2}} + \frac{-1}{\sqrt{x_p^2+y_p^2+\left(z_p+6d\right)^2}} \\ +\ldots \end{array}\right] = kq\left[\frac{1}{\sqrt{x_p^2+y_p^2+z_p^2}} + \sum_{i=1}^{\infty}\frac{(-1)^i}{\sqrt{x_p^2+y_p^2+\left(z_p-2id\right)^2}} + \sum_{i=1}^{\infty}\frac{(-1)^i}{\sqrt{x_p^2+y_p^2+\left(z_p+2id\right)^2}}\right]$$

This is in the region between the two planes.

You can, pretty easily, find the potential at a particular point (such as z=d/2, x=0, y=0):

$$V\left(\vec{r}_p\right) = kq\left[\frac{1}{|z_p|} + \sum_{i=1}^{\infty}\frac{(-1)^i}{|z_p-2id|} + \sum_{i=1}^{\infty}\frac{(-1)^i}{|z_p+2id|}\right]$$

If you work this problem, you'll probably want to look up the series in a reference book somewhere to recognize the sequence.

This does converge since it falls off faster than the "p" series.

Also there is a warning about a numerical summation here: It will probably not give very good results since the errors will tend to be additive. You would therefore really want to find a method for summation of the series in order to use them.