Problem 2.7: find the electric field a distance z from a spherical shell of charge.

$$\vec{E} = \int k\,dq \frac{-x\hat{x} - y\hat{y} + (z_p - z)\hat{z}}{\left(x^2 + y^2 + (z_p - z)\right)^{3/2}}$$

$$\int_{\varphi=0}^{\varphi=2\pi} \int_{\theta=0}^{\theta=\pi} \sigma R^2 \sin(\theta) d\theta d\varphi = Q \Rightarrow \sigma R^2 \sin(\theta) d\theta d\varphi = dq$$

$$\vec{E} = \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} k \frac{-x\hat{x} - y\hat{y} + (z_p - z)\hat{z}}{\left(x^2 + y^2 + (z_p - z)\right)^{3/2}} \sigma R^2 \sin(\theta) d\theta d\varphi$$

we can convert to spherical polar coordinates:

$$\vec{E} = \int_{\theta=0}^{\theta=\pi} \int_{\varphi=0}^{\varphi=2\pi} k \frac{z_p\hat{z} - R\hat{r}}{\left|z_p\hat{z} - R\hat{r}\right|^3} \sigma R^2 \sin(\theta) d\theta d\varphi$$

We can write

$$z_p\hat{z} - R\hat{r} = -R\sin(\theta)\cos(\varphi)\,\hat{x} - R\sin(\theta)\sin(\varphi)\,\hat{y} + \left(z_p - R\cos(\theta)\right)\hat{z}$$

$$\left|z_p\hat{z} - R\hat{r}\right| = \sqrt{R^2\sin^2(\theta) + \left(z_p - R\cos(\theta)\right)^2} = \sqrt{R^2 - 2Rz_p\cos(\theta) + z_p^2}$$

Let's try to evaluate these integrals:

$$\vec{E} = 2\pi\sigma R^2 \int_{\theta=0}^{\theta=\pi} k \frac{\left(z_p - R\cos(\theta)\right)\hat{z}}{\left(R^2 - 2Rz_p\cos(\theta) + z_p^2\right)^{3/2}} \sin(\theta) d\theta$$

(the x and y parts vanish in the ϕ integration)

let $y \equiv z_p - R\cos(\theta) \Rightarrow dy = R\sin(\theta)d\theta; \theta = 0 \Rightarrow y = z_p - R : \theta = \pi \Rightarrow y = z_p + R$

$$2yz_p - 2z_p^2 = -2Rz_p\cos(\theta)$$

$$\vec{E} = 2\pi\sigma R \int_{y=z_p-R}^{y=z_p+R} k \frac{y\,dy}{\left(R^2 + 2yz_p - z_p^2\right)^{3/2}} \hat{z} = 2\pi\sigma k R \left[\sqrt{R^2 + 2yz_p - z_p^2}\left(\frac{2}{4z_p^2} + \frac{2\left(R^2 - z_p^2\right)}{4z_p^2\left(R^2 + 2yz_p - z_p^2\right)}\right)\right]_{y=z_p-R}^{y=z_p+R} \hat{z}$$

$$\vec{E} = \frac{\pi\sigma k R}{z_p^2}\left[\sqrt{R^2 + 2yz_p - z_p^2}\left(1 + \frac{\left(R^2 - z_p^2\right)}{\left(R^2 + 2yz_p - z_p^2\right)}\right)\right]_{y=z_p-R}^{y=z_p+R} \hat{z}$$

$$\vec{E} = \frac{\pi\sigma k R}{z_p^2}\left[\sqrt{R^2 + 2yz_p - z_p^2}\left(\frac{\left(R^2 + 2yz_p - z_p^2\right) + \left(R^2 - z_p^2\right)}{\left(R^2 + 2yz_p - z_p^2\right)}\right)\right]_{y=z_p-R}^{y=z_p+R} \hat{z}$$

$$\vec{E} = \frac{\pi\sigma k R}{z_p^2}\left[\sqrt{R^2 + 2yz_p - z_p^2}\left(\frac{\left(R^2 + 2yz_p - z_p^2\right) + \left(R^2 - z_p^2\right)}{\left(R^2 + 2yz_p - z_p^2\right)}\right)\right]_{y=z_p-R}^{y=z_p+R} \hat{z}$$

$$\vec{E} = \frac{2\pi\sigma k R}{z_p^2}\left[\left(\frac{\left(R^2 + yz_p - z_p^2\right)}{\sqrt{\left(R^2 + 2yz_p - z_p^2\right)}}\right)\right]_{y=z_p-R}^{y=z_p+R} \hat{z}$$

$$\vec{E} = \frac{2\pi\sigma kR}{z_p^2}\left[\left(\frac{\left(R^2+\left(z_p+R\right)z_p-z_p^2\right)}{\sqrt{\left(R^2+2\left(z_p+R\right)z_p-z_p^2\right)}} - \frac{\left(R^2+\left(z_p-R\right)z_p-z_p^2\right)}{\sqrt{\left(R^2+2\left(z_p-R\right)z_p-z_p^2\right)}}\right)\right]\hat{z}$$

$$\vec{E} = \frac{2\pi\sigma kR}{z_p^2}\left[\left(\frac{\left(R^2+\left(z_p+R\right)z_p-z_p^2\right)}{\sqrt{\left(R^2+2Rz_p+z_p^2\right)}} - \frac{\left(R^2+\left(z_p-R\right)z_p-z_p^2\right)}{\sqrt{\left(R^2-2Rz_p+z_p^2\right)}}\right)\right]\hat{z}$$

$$\vec{E} = \frac{2\pi\sigma kR}{z_p^2}\left[\left(\frac{\left(R^2+Rz_p\right)}{\sqrt{\left(R^2+2Rz_p+z_p^2\right)}} - \frac{\left(R^2-Rz_p\right)}{\sqrt{\left(R^2-2Rz_p+z_p^2\right)}}\right)\right]\hat{z}$$

$$\vec{E} = \frac{2\pi\sigma kR^2}{z_p^2}\left[\left(\frac{\left(R+z_p\right)}{\sqrt{\left(R^2+2Rz_p+z_p^2\right)}} - \frac{\left(R-z_p\right)}{\sqrt{\left(R^2-2Rz_p+z_p^2\right)}}\right)\right]\hat{z}$$

$$\vec{E} = \frac{2\pi\sigma kR^2}{z_p^2}\left[\left(\frac{\left(R+z_p\right)}{\sqrt{\left(R+z_p\right)^2}} - \frac{\left(R-z_p\right)}{\sqrt{\left(R-z_p\right)^2}}\right)\right]\hat{z}$$

Case 1: $R>z_p$ (let's keep $z_p >=0$)

$$R+z_p = R+z_p$$

$$\left|R-z_p\right| = R-z_p$$

$$\vec{E} = \frac{2\pi\sigma kR^2}{z_p^2}\left[\left(1-1\right)\right]\hat{z} = 0\hat{z}$$

Case 2: $R<z_p$

$$R+z_p = R+z_p$$

$$\left|R-z_p\right| = z_p - R$$

$$\vec{E} = \frac{2\pi\sigma kR^2}{z_p^2}\left[\left(1-\frac{\left(R-z_p\right)}{z_p-R}\right)\right]\hat{z} = \frac{2\pi\sigma kR^2}{z_p^2}\left[\left(1+1\right)\right]\hat{z} = \frac{4\pi\sigma kR^2}{z_p^2}\hat{z}$$

Now how much charge is enclosed for this case?

$$Q = 4\pi R^2\sigma$$

$$\vec{E} = k\frac{Q}{z_p^2}\hat{z}$$

Problem 2.8

What about a volume charge distribution, ρ?

From the results of the above problem, we've shown that the only contribution to the electric field comes from those charges that are at $R<z_p$. Invariably, the integral over the volume charge distribution will then exclude charges outside z.

The electric field arising from charges less than z_p is given by

$$d\vec{E} = \frac{4\pi\rho kr^2}{z_p^2}dr\hat{z}$$

Thus, so long as $z_p<R$,

$$\vec{E} = \int_{r=0}^{z_p} d\vec{E} = \int_{r=0}^{z_p}\frac{4\pi\rho kr^2}{z_p^2}dr\hat{z} = k\frac{4}{3}\frac{\pi\rho r^3}{z_p^2}\hat{z}\Big|_{r=0}^{r=z_p} = k\frac{4}{3}\pi\rho z_p\hat{z} = k\frac{Q}{R^3}z_p\hat{z}$$

if $z_p > R$, then

$$\vec{E} = \int_{r=0}^{R} d\vec{E} = \int_{r=0}^{R} \frac{4\pi\rho k r^2}{z_p^2} dr\hat{z} = k\frac{4}{3}\frac{\pi\rho r^3}{z_p^2}\hat{z}\Big|_{r=0}^{r=R} = k\frac{4}{3}\pi\rho\frac{R^3}{z_p^2}\hat{z} = k\frac{Q}{z_p^2}\hat{z}$$

0.1	1
0.2	2
0.3	3
0.4	4
0.5	5
0.6	6
0.7	7
0.8	8
0.9	9
1	10
0.826446	11
0.694444	12
0.591716	13
0.510204	14
0.444444	15
0.390625	16
0.346021	17
0.308642	18
0.277008	19
0.25	20
0.226757	21
0.206612	22
0.189036	23
0.173611	24
0.16	25
0.147929	26
0.137174	27
0.127551	28
0.118906	29