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$$\vec{v} = r^2 \cos \theta \hat{r} + r^2 \cos \varphi \hat{\theta} - r^2 \cos \theta \sin \varphi \hat{\phi}$$

Divergence Theorem:

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{v}) d\tau = \oint_{\text{surface}} \vec{v} \cdot d\vec{A}$$

Surface: positive octant of a sphere.

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin(\theta)} \frac{\partial}{\partial \theta} (\sin(\theta) v_\theta) + \frac{1}{r \sin(\theta)} \frac{\partial v_\phi}{\partial \phi}$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} [r^4 \cos \theta] = \frac{4}{r^2} r^3 \cos \theta = 4r \cos \theta$$

$$\frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} [\sin \theta r^2 \cos \varphi] = r \cot \theta \cos \varphi$$

$$-\frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} [r^2 \cos \theta \sin \varphi] = -r \cot \theta \cos \varphi$$

$$\Rightarrow \vec{\nabla} \cdot \vec{v} = 4r \cos \theta$$

$$\oint 4r \cos \theta d\tau = \int_{r=0}^a \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} 4r \cos \theta r^2 \sin(\theta) dr d\theta d\phi = a^4 \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{\pi/2} \cos \theta \sin(\theta) d\theta d\phi = a^4 \frac{\pi}{2} \int_{\theta=0}^{\pi/2} \sin(\theta) d[\sin \theta]$$

$$= a^4 \frac{\pi}{2} \left[\frac{1}{2} \sin^2 \theta \right]_0^{\pi/2} = \frac{\pi a^4}{4}$$

$$\oint_{\text{sphere}} \vec{v} \cdot d\vec{A} = \int \vec{v} \cdot r^2 \sin \theta d\theta d\phi \hat{r}$$

$$+ \int_{\text{disk } xz} \vec{v} \cdot (-\hat{y}) dx dz + \int_{\text{disk } xy} \vec{v} \cdot (-\hat{z}) dx dy + \int_{\text{disk } yz} \vec{v} \cdot (-\hat{x}) dy dz$$

Spherical unit vectors are given by:

$$\hat{r} = \sin(\theta) \cos(\varphi) \hat{x} + \sin(\theta) \sin(\varphi) \hat{y} + \cos(\theta) \hat{z}$$

$$\hat{\theta} = \cos(\theta) \cos(\varphi) \hat{x} + \cos(\theta) \sin(\varphi) \hat{y} - \sin(\theta) \hat{z}$$

$$\hat{\phi} = -\sin(\varphi) \hat{x} + \cos(\varphi) \hat{y}$$

$$\hat{r} \cdot \hat{x} = \sin(\theta) \cos(\varphi) : \hat{\theta} \cdot \hat{x} = \cos(\theta) \cos(\varphi) : \hat{\phi} \cdot \hat{x} = -\sin(\varphi)$$

$$\vec{v} \cdot \hat{x} = r^2 \cos \theta \sin(\theta) \cos(\varphi) + r^2 \cos \varphi \cos(\theta) \cos(\varphi) + r^2 \cos \theta \sin \varphi \sin(\varphi)$$

$$= r^2 \cos \theta \sin(\theta) \cos(\varphi) + r^2 \cos \theta$$

$$\text{In } yz : \varphi = \frac{\pi}{2} \Rightarrow -\vec{v} \cdot \hat{x} = -r^2 \cos \theta$$

$$\hat{r} \cdot \hat{y} = \sin(\theta) \sin(\varphi) : \hat{\theta} \cdot \hat{y} = \cos(\theta) \sin(\varphi) : \hat{\phi} \cdot \hat{y} = \cos(\varphi)$$

$$\vec{v} \cdot \hat{y} = \vec{v} = r^2 \cos \theta \sin(\theta) \sin(\varphi) + r^2 \cos \varphi \cos(\theta) \sin(\varphi) - r^2 \cos \theta \sin \varphi \cos(\varphi) = r^2 \cos \theta \sin(\theta) \sin(\varphi)$$

$$\text{in } xz \text{ plane} : \varphi = 0 \Rightarrow \vec{v} \cdot \hat{y} = 0$$

$$\hat{r} \cdot \hat{z} = \cos(\theta) : \hat{\theta} \cdot \hat{z} = -\sin(\theta) : \hat{\phi} \cdot \hat{z} = 0$$

$$\vec{v} \cdot \hat{z} = r^2 \cos \theta \cos(\theta) - r^2 \cos \varphi \sin(\theta)$$

$$\text{in } xy \text{ plane} : \theta = \frac{\pi}{2} \Rightarrow -\vec{v} \cdot \hat{z} = r^2 \cos \varphi$$

on sphere : $\vec{v} \cdot \hat{r} = r^2 \cos \theta$

Form integrals:

$$\int_{\text{sphere}} \vec{v} \cdot d\vec{A} = \int_{\varphi=0}^{\varphi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=\frac{\pi}{2}} r^2 \cos \theta r^2 \sin \theta d\theta d\varphi = r^4 \frac{\pi}{2} \frac{1}{2} \sin^2 \theta \Big|_0^{\pi/2} = r^4 \frac{\pi}{4}$$

Integral in xz plane vanished. We don't need to do the integrals in the xy and yz planes since they both have exactly the same form but are opposite in sign. Thus the divergence theorem is verified.