

Problem 2.11

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0}$$

Inside the sphere, by symmetry E will vanish. This is insured by the uniform charge density. I believe we can also show this by coulomb's law:

$$\vec{r}_i = x_i \hat{x} + y_i \hat{y} + z_i \hat{z} = r \sin \theta \cos \phi \hat{x} + r \sin \theta \sin \phi \hat{y} + r \cos \theta \hat{z}$$

$$\vec{r}_p = z_p \hat{z} \Rightarrow \vec{r}_{ip} = -r \sin \theta \cos \phi \hat{x} - r \sin \theta \sin \phi \hat{y} + (z_p - r \cos \theta) \hat{z}$$

$$dq_i = \sigma a^2 \sin \theta d\theta d\phi$$

$$\vec{E} = \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} k\sigma a^2 \frac{-r \sin \theta \cos \phi \hat{x} - r \sin \theta \sin \phi \hat{y} + (z_p - r \cos \theta) \hat{z}}{[r^2 \sin^2 \theta + (z_p - r \cos \theta)^2]^{\frac{3}{2}}} \sin \theta d\theta d\phi$$

The x and y integrals vanish by symmetry. Thus:

$$\vec{E} = 2\pi k\sigma a^2 \int_{\theta=0}^{\theta=\pi} \frac{(z_p - r \cos \theta) \hat{z}}{[r^2 - 2rz_p \cos \theta + z_p^2]^{3/2}} \sin \theta d\theta \hat{z} = 2\pi k\sigma a^2 \left[\frac{r - z_p \cos \theta}{z_p^2 \sqrt{r^2 - 2rz_p \cos \theta + z_p^2}} \right]_{\theta=0}^{\theta=\pi} \hat{z} = 2\pi k\sigma a^2 \left\{ \frac{r+z_p}{z_p^2 \sqrt{r^2 + 2rz_p + z_p^2}} - \frac{r-z_p}{z_p^2 \sqrt{r^2 - 2rz_p + z_p^2}} \right\} \hat{z}$$

$$\vec{E} = 2\pi k\sigma a^2 \left\{ \frac{r+z_p}{z_p^2 \sqrt{(r+z_p)^2}} - \frac{r-z_p}{z_p^2 \sqrt{(r-z_p)^2}} \right\} \hat{z}$$

keeping z positive, inside the sphere, this becomes:

$$\vec{E} = 2\pi k\sigma a^2 \left\{ \frac{1}{z_p^2} - \frac{1}{z_p^2} \right\} \hat{z} = \vec{0}$$

Outside the sphere, we have:

$$E(4\pi r^2) = \frac{\sigma(4\pi a^2)}{\epsilon_0} \Rightarrow \vec{E} = \frac{\sigma a^2}{\epsilon_0 r^2} \hat{r}$$

Notice the discontinuity at the surface. This would also happen if the shell were a conductor. Basically, the shell of charge is the same that would result from a solid conductor. However, unlike a conductor, it would still be possible to impose an external field on this problem with the consequence of a non-zero electric field inside the shell.