

Section 1.5: The Dirac Delta Function

We'll consider the vector function:

$$\vec{V} = \frac{1}{r^2} \hat{r}$$

Look at the divergence of this function:

$$\vec{\nabla} \cdot \vec{V} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 V_r) = \frac{1}{r^2} \frac{\partial(1)}{\partial r} = 0$$

(the function has 0 divergence)

Now, look at the integral over the surface of a sphere of radius R:

$$\oint \vec{V} \cdot d\vec{A} = \int \left(\frac{1}{R^2} \hat{r} \right) \cdot (R^2 \sin(\theta) d\theta d\phi \hat{r}) = 4\pi$$

Let's look at this in light of the divergence theorem:

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{V}) d\tau = \oint_{\text{surface}} \vec{V} \cdot d\vec{A}$$

Normally, you'd think that if $\vec{\nabla} \cdot \vec{V} = 0$ then we also have:

$$\int_{\text{volume}} (\vec{\nabla} \cdot \vec{V}) d\tau = 0$$

But this particular function clearly can't have this be the case.

Where the problem is is at the origin, where the function seems to diverge.

Your author states that the divergence of this function is central to all of E&M.

I'm not completely sure he's correct on that point but I'll move on now.

Here is the important point which makes it necessary to look at the delta function:

$$\vec{\nabla} \cdot \left[\frac{\hat{r}}{r^2} \right] = 4\pi \delta^3(\vec{r})$$

The 1-d Dirac Delta Function:

$$\delta(x) = \begin{cases} 0, & \text{if } x \neq 0 \\ \infty, & \text{if } x=0 \end{cases}; \int_{-\infty}^{+\infty} \delta(x) dx = 1$$

$$\delta(x-a) = \begin{cases} 0, & \text{if } x \neq a \\ \infty, & \text{if } x=a \end{cases}; \int_{-\infty}^{+\infty} \delta(x-a) dx = 1$$

For 2 functions involving the delta function ($D_1(x)$ and $D_2(x)$):

$$\text{If: } \int_{-\infty}^{+\infty} f(x) D_1(x) dx = \int_{-\infty}^{+\infty} f(x) D_2(x) dx$$

$$\text{Then: } D_1(x) = D_2(x)$$

A very closely related function is the Heviside Step function.

$$\theta(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}; \frac{d\theta}{dx} = \delta(x)$$

How do you use this?

$$f(x) \delta(x) = f(0) \delta(x) \text{ and } f(x) \delta(x-a) = f(a) \delta(x-a)$$

$$\int_{-\infty}^{+\infty} f(x) \delta(x) dx = f(0) \int_{-\infty}^{+\infty} \delta(x) dx = f(0) \text{ and } \int_{-\infty}^{+\infty} f(x) \delta(x-a) dx = f(a) \int_{-\infty}^{+\infty} \delta(x-a) dx = f(a)$$

$$\delta(kx) = \frac{1}{|k|} \delta(x) \text{ (example 1.15) } \quad x \frac{d(\delta(x))}{dx} = -\delta(x) \text{ (problem 1.45)}$$

The Three-Dimensional Delta Function (1.5.3)

$$\int_{\text{all space}} \delta^3(\vec{r}) d\tau \equiv \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \delta(\bar{x})\delta(\bar{y})\delta(\bar{z}) dx dy dz = 1$$
$$\int_{\text{all space}} f(\vec{r}) \delta^3(\vec{r} - \vec{a}) d\tau = f(\vec{a})$$

Back to the earlier divergence problem. The upshot is this:

$$\vec{\nabla} \cdot \left(\frac{\hat{r}}{r^2} \right) = 4\pi \delta^3(\vec{r})$$

and more generally,

$$\vec{\nabla} \cdot \left(\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2} \right) = 4\pi \delta^3(\vec{\mathfrak{R}})$$

The author makes an important point: the differentiation is w/r to r!

Also, since:

$$\vec{\nabla} \left(\frac{1}{\mathfrak{R}} \right) = -\frac{\hat{\mathfrak{R}}}{\mathfrak{R}^2}$$

we have:

$$\vec{\nabla}^2 \frac{1}{\mathfrak{R}} = -4\pi \delta^3(\vec{\mathfrak{R}}) \text{ (problem 1.13)}$$

The Theory of Vector Fields

to what extent is a vector function determined by its divergence and curl?

We'll let \vec{F} stand for \vec{E} or \vec{B} here.

Suppose:

$$\vec{\nabla} \cdot \vec{F} = D \text{ and } \vec{\nabla} \times \vec{F} = \vec{C} \text{ where } \vec{\nabla} \cdot \vec{C} = 0$$

(divergence of a curl is always zero)

The question is: can you determine \vec{F} uniquely?

The answer is yes, provided that you include boundary conditions.
(this is the basis of the Helmholtz Theorem, proven in Appendix B)

Now we've got a very important point:

$$\vec{\nabla} \times \vec{F} = 0 \Leftrightarrow \vec{F} = -\vec{\nabla} V$$

(- is the convention here)

V is a scalar potential as opposed to \vec{A} which will be a vector potential:

$$\vec{\nabla} \cdot \vec{F} = 0 \Leftrightarrow \vec{F} = \vec{\nabla} \times \vec{A}$$

For a more general vector, no matter what its divergence and curl are,

$$\vec{F} = -\vec{\nabla} V + \vec{\nabla} \times \vec{A}$$

I'd like for you to look at page 419 in your text.

We can modify V and \vec{A} by the following:

$$\vec{A}' = \vec{A} + \vec{\nabla} \lambda$$

$$V' = V - \frac{\partial \lambda}{\partial t}$$

will leave \vec{E} and \vec{B} unchanged. This is called a Gauge Transformation.

2 historically important gauges:

Lorentz Gauge:

$$\vec{\nabla} \cdot \vec{A} = -\mu_0 \epsilon_0 \frac{\partial V}{\partial t}$$

Coulomb Gauge:

$$\vec{\nabla} \cdot \vec{A} = 0$$

Provided we get to chapter 10, we'll come back to these.

Class Problems: 1.49 (a),(b)

1.53

1.54

1.56

1.58