Physics Version

Spherical Coordinates: Gravitational Attraction

The transformation from Cartesian to Spherical Coordinates is:

$$x = r\sin\theta\cos\varphi$$

$$y = r\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

We form the Jacobian:

$$\left|\vec{J}\right| = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\left|\vec{J}\right| = \begin{bmatrix} \sin\theta\cos\varphi\left[r^2\sin^2\theta\cos\varphi\right] \\ -r\cos\theta\cos\varphi\left[-r\sin\theta\cos\theta\cos\varphi\right] = r^2\left[\cos^2\varphi\sin\theta\right] + r^2\sin\theta\sin^2\varphi \\ r^2\sin\theta\sin^2\varphi \end{bmatrix} = r^2\sin\theta$$

This gives us the linear transformation :

$$dx\,dy\,dz \to r\sin\theta\,dr\,d\varphi\,d\theta$$

Example 1: Find the volume of the sphere:

$$x^2 + y^2 + z^2 = a^2$$

The integral is given by:

$$V = \int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi}\int_{r=0}^{r=a} r^2\sin\theta\,dr\,d\varphi\,d\theta = 2\pi\left[\tfrac{a^3}{3}\right]\left[-\cos\theta\right]\Big|_0^{\pi} = -2\pi\left[\tfrac{a^3}{3}\right]\left[-1-1\right] = \tfrac{4}{3}\pi a^3$$

be careful about the azimuthal limits so that you don't add up your sphere twice too many times.

Example 2

Find the centroid of the region bounded by the sphere $r = a$ and the cone $\theta = \alpha$

The volume of the solid is:

$$V = \int_{\theta=0}^{\theta=\alpha}\int_{r=0}^{r=a}\int_{\varphi=0}^{\varphi=2\pi} r^2\sin\theta\,d\theta\,dr\,d\varphi = 2\pi\tfrac{a^3}{3}\left[-\cos\theta\right]\Big|_0^{\alpha} = \tfrac{2\pi a^3}{3}\left[1-\cos\alpha\right]$$

The z coordinate of the centroid is given by:

$$\overline{z} = \tfrac{1}{V}\int_{\theta=0}^{\theta=\alpha}\int_{r=0}^{r=a}\int_{\varphi=0}^{\varphi=2\pi} zr^2\sin\theta\,d\theta\,dr\,d\varphi = \tfrac{3}{a^3[1-\cos\alpha]}\int_{\theta=0}^{\theta=\alpha}\tfrac{a^4}{4}\cos\theta\sin\theta\,d\theta = \tfrac{3a}{4[1-\cos\alpha]}\int_{\theta=0}^{\theta=\alpha}\sin\theta\left[d\left(\sin\theta\right)\right]$$

$$= \tfrac{3a}{4[1-\cos\alpha]}\tfrac{\sin^2\theta}{2}\Big|_0^{\alpha} = \tfrac{3\sin^2\alpha}{8[1-\cos\alpha]}a = \tfrac{3}{8}a\tfrac{[1-\cos\alpha][1+\cos\alpha]}{[1-\cos\alpha]} = \tfrac{3}{8}a\left[1+\cos\alpha\right]$$

Example 3: Done the right way.

The vector pointing to a differential mass element

$dm = \sigma r^2\sin\theta\,dr\,d\theta\,d\varphi$ is given by:

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

where the Cartesian coordinates are the points of a mass point relative to the origin.

The vector pointing to a point located along the z-axis is given by:

$$\vec{r}_p = 0\hat{i} + 0\hat{j} + z_p\hat{k}$$

The gravitational attraction for a mass element and a mass m located at the point is given by:

$$d\vec{F} = G\frac{m(dm_i)}{\left[\sqrt{x_i^2+y_i^2+(z_p-z_i)^2}\right]^2}\left[\hat{r}_{ip}\right] = -G\frac{m(dm_i)}{\left[\sqrt{x_i^2+y_i^2+(z_p-z_i)^2}\right]^2}\frac{x_i\hat{i}+y_i\hat{j}+(z_i-z_p)\hat{k}}{\sqrt{x_i^2+y_i^2+(z_p-z_i)^2}} = -mG\frac{x_i\hat{i}+y_i\hat{j}+(z_i-z_p)\hat{k}}{\left[x_i^2+y_i^2+(z_p-z_i)^2\right]^{\frac{3}{2}}}(dm_i)$$

We are going to integrate this over a sphere of radius a. I'll worry about scaling the mass later to fit this.

$$\vec{F} = \int d\vec{F} = -m\sigma G\int_{\theta=0}^{\theta=\pi}\int_{r=0}^{r=a}\int_{\varphi=0}^{\varphi=2\pi}\frac{x_i\hat{i}+y_i\hat{j}+(z_i-z_p)\hat{k}}{\left[x_i^2+y_i^2+(z_p-z_i)^2\right]^{\frac{3}{2}}}r^2\sin\theta\,dr\,d\theta d\varphi$$

For the rest of the solution to this problem, I'll refer you to a similar problem which appears on my pages. You may investigate the details there if you wish. Soon there will be a much easier way to work this type of problem!

Look at the link entitled “sphere of charge” for more information. We will get there soon.

In that problem, the analysis is quite similar, only the constants are different.