Wave Motion

Simplest form of a sinusoidal wave:

$$\Psi(x,t)|_{t=0}=A\sin\left(\frac{2\pi}{\lambda}x\right)$$

which represents the instantaneous amplitude or displacement of a wave.
As time increases, if the wave travels to the right, the wave is given by:

$$\Psi(x,t)=A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

λ is the wavelength,
v is the velocity (actually called the phase velocity) of the wave.
The period T is the time required for the wave to travel a distance of one wavelength.
We also have the connection between wavelength, frequency and velocity:

$$\lambda=vT=\frac{v}{f}\Rightarrow f\lambda=v$$

A traveling wave satisfies the wave equation:

$$\frac{\partial^2\Psi}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}$$

Let me show this for our wave above:

$$\frac{\partial\Psi}{\partial x}=\frac{2\pi}{\lambda}A\cos\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

$$\frac{\partial^2\Psi}{\partial x^2}=-\left[\frac{2\pi}{\lambda}\right]^2A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

$$\frac{\partial\Psi}{\partial t}=-\left[\frac{2\pi}{\lambda}\right]vA\cos\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

$$\frac{\partial^2\Psi}{\partial t^2}=-\left[\frac{2\pi}{\lambda}\right]^2v^2A\sin\left[\frac{2\pi}{\lambda}(x-vt)\right]$$

Thus you can see for our solution above:

$$\frac{\partial^2\Psi}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2\Psi}{\partial t^2}\quad.$$

The equation is satisfied by our sinusoidal wave.

It is normally helpful to rewrite the wave form as:

$$\Psi(x,t)=A\sin\left[2\pi\left(\frac{x}{\lambda}-\frac{t}{T}\right)\right]$$

which shows the true dimensionless form of the argument.
We now normally define a wave number and an angular frequency by:

$$k\equiv\frac{2\pi}{\lambda};\omega=\frac{2\pi}{T}$$

Then the wave form becomes:

$$\Psi(x,t) = A\sin[kx - \omega t]$$

A wave traveling in the –x direction appears as:

$$\Psi(x,t) = A\sin[kx + \omega t]$$

The phase velocity is the velocity of a point on the wave (the crest, for example) and is given by:

$$v_{ph} = \frac{\lambda}{T} = \frac{\omega}{k}$$

We can also introduce a phase so that the amplitude need not be zero at t=0:

$$\Psi(x,t) = A\sin[kx - \omega t + \phi]$$

With the correct choice of phase, the wave may be expressed as a cosine function:

$$\Psi(x,t) = A\cos[kx - \omega t]$$

when the phase is 90 degrees.

Waves obey the principle of superposition: the displacement in a region of space comes from adding the displacements of all the waves.

Let's see how this happens. Suppose you have two waves:

$$\Psi_1(x,t) = A\cos[k_1 x - \omega_1 t]$$

$$\Psi_2(x,t) = A\cos[k_2 x - \omega_2 t]$$

We can add these to obtain using obtain using:

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

See: http://mathworld.wolfram.com/TrigonometricAdditionFormulas.html

$$\Psi(x,t) = \Psi_1(x,t) + \Psi_2(x,t) = A\cos[k_1 x - \omega_1 t] + A\cos[k_2 x - \omega_2 t]$$

$$= 2A\cos\left[\frac{1}{2}(k_1 - k_2)x - \frac{1}{2}(\omega_1 - \omega_2)t\right]\cos\left[\frac{1}{2}(k_1 + k_2)x - \frac{1}{2}(\omega_1 + \omega_2)t\right]$$

$$= 2A\cos\left[\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right]\cos[\bar{k}x - \bar{\omega}t]$$

Where $\Delta k = k_1 - k_2; \Delta\omega = \omega_1 - \omega_2; \bar{k} = \frac{1}{2}(k_1 + k_2)\bar{\omega} = \frac{1}{e}(\omega_1 + \omega_2)$

There is an animation of this on the home page in terms of beat formation. Superposition in terms of sound is also presented there.

We identify a localized region of space defined by two consecutive points where the envelope is zero:

$$\Delta x = x_2 - x_1$$

Then:

$$\frac{1}{2}\Delta k\ x_2 - \frac{1}{2}\Delta k\ x_1 = \pi \Rightarrow \Delta k \Delta x = 2\pi \Rightarrow \Delta\omega\Delta t = 2\pi$$

Then:

This is the case for 2 sinusoidal waves.

If you have many waves, you would do the Fourier analysis of the wave:

$$\Psi(x,t) = \sum_i A_i \cos(k_i x - \omega_i t)$$

If the amplitudes are continuous, then the generalization of this is the Fourier Integral:

$$\Psi(x,t) = \int_{\text{all } k} \tilde{A}(k')\cos(k' x - \omega(k')t)\,dk'$$

You might want to know what is the tightest that you can make a wave packet? This answer is given by the form of a Gaussian wave packet:

$$\Psi(x,0) = \Psi(x) = A\,e^{-\Delta k^2 \Delta x^2}\cos(k_0 x)$$

Roughly, the value of the wave form at $x = \Delta x/2$ is 0.6

or:

$$e^{-\Delta k^2 \Delta x^2/4} \approx 0.6 \Rightarrow -\frac{\Delta k^2 \Delta x^2}{4} = -0.5 \Rightarrow \Delta k \Delta x \approx 1.4$$

A more precise analysis would yield the result of ½.

Let's just look at the Gaussian distribution now. A normal distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{\pi}} e^{-\left[\frac{(x-x_0)^2}{2\sigma^2}\right]}$$

where σ is the standard deviation given by: $\sigma = \sqrt{\sum_{i=1}^{i=N} p_i (x_i - \mu_i)^2}; \mu_i = \sum_{i=1}^{i=N} p_i x_i$

For a continuous variable, $\sigma = \sqrt{\int p(x)(x-\mu)^2 dx}; \mu = \int x\,p(x)\,dx$

At full width, half maximum, the relationship is:

$$\frac{1}{2} = e^{-\left[\frac{(x-x_0)^2}{2\sigma^2}\right]} \Rightarrow -\ln(2) = -\left[\frac{(x-x_0)^2}{2\sigma^2}\right]$$

let $x = x_0$ (the point at half-width, full maximum); then the peak is at μ.

This then leads to the width in x being given by 0.235σ.

There is another type of velocity that is important to discuss, namely the group velocity.

$$v_g = \left(\frac{d\omega}{dk}\right)_{k=k_0}$$

For a de Broglie wave, we have E=hf and p=h/λ.

These equations can be rewritten as:

$$E = hf = \frac{h}{2\pi}(2\pi f) = \hbar\omega$$

$$p = \frac{h}{\lambda} = \hbar\left(\frac{2\pi}{\lambda}\right) = \hbar k$$

where I've used: $\omega = 2\pi f; k = \frac{2\pi}{\lambda}$.

Looking back at the definition for the group velocity, we have:

$$v_g = \left(\frac{d\omega}{dk}\right)_{k=k_0} = \left(\frac{d(\hbar\omega)}{d(\hbar k)}\right)_{\hbar k = \hbar k_0} = \left(\frac{dE}{dp}\right)_{E=E_0}$$

For matter, we have:

$$E^2 = p^2 c^2 + m^2 c^4 \Rightarrow 2E\, dE = 2pc^2\, dp \Rightarrow \frac{dE}{dp} = \frac{pc^2}{E} \quad .$$

This is the velocity of a particle of momentum p and total energy E. It is therefore reasonable to assume that the group velocity of a wave packet can be associated with the velocity of a particle.

The phase velocity of the wave is given by

$$v_{ph} = f\lambda = \frac{\omega}{k} \Rightarrow \omega = k v_{ph}$$

The group velocity is thus related also to the phase velocity by:

$$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(v_{ph} k) = v_{ph} + k\frac{dv_{ph}}{dk}$$

Thus, the group velocity may be greater or less then the phase velocity. A medium is called "non-dispersive" when the phase velocity is the same for all frequencies and $v_g = v_{ph}$. Electromagnetic waves in a vacuum provide just such an example. Water waves, on the other hand, are dispersive. Throwing a rock into water demonstrates that the envelope of the wave moves more slowly than the individual waves moving outward.

Dispersion acting on tightly packed wave packets forces the packets to significantly distort with time.

Example: According to Newton, deep water waves have a phase velocity of $v_{ph} = \sqrt{\frac{\lambda g}{2\pi}}$. Find the group velocity of such waves and discuss the motion.

$$v_{pk} = \sqrt{g}\, k^{-1/2} \Rightarrow v_g = \sqrt{\frac{g}{k}} + k\, d\frac{\left[\sqrt{g}\, k^{-1/2}\right]}{dk} = \frac{1}{2} v_{ph}$$

According to your author, as the radial waves move out, the individual waves seem to run right through the wave crests and then disappear (after throwing a rock into the water).