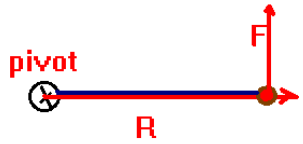
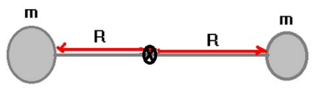
r20	Physics 240: Worksheet 18	Name _________________

(1) A force F acts on a point mass which is held by a massless rigid rod at a distance R from a pivot point and at an angle of 90°. Describe what happens.



(2) A dumbbell consists of 2 point masses connected by a massless rod and is pivoted about its center of mass as shown below. Calculate the moment of inertia of the system about the center of mass.



(3) A point mass m is connected by a massless rod at a distance R from an infinitely strong pivot. A force F is applied to the mass at right angles. Describe what happens. The force is then removed. Describe what happens.

(4) A point mass m is connected by a massless rod at a distance R from an infinitely strong pivot. The mass is moving with a tangential velocity v. What is the kinetic energy of the system?

(5) Showing details, calculate the moment of inertia for a solid sphere of radius b with a constant volume mass density ρ. Then calculate the moment of inertia for two concentric thin shelled cylinders of radius a and b with masses m_a and m_b about the symmetry axis. Then calculate the moment of inertia for a solid disk of radius r with a point mass m at a distance b from the center of the disk about the perpendicular axis through the center of the disk.

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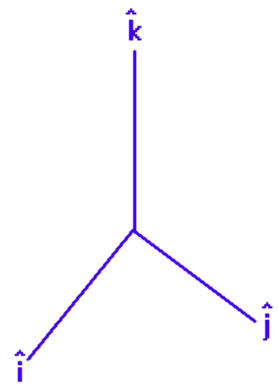
The Right Hand Rule for Torque (RHR#1)

The validity of any right hand rule lies in it's mathematical import. The fundamental thing behind the definition of torque is this:

$$\vec{\Gamma} = \vec{R} \times \vec{F}$$

which is the **vector cross product** of R and F. To understand how to develop a right hand rule, look at the ***right hand coordinate*** system shown below:



This is just the x-y-z coordinate system. The rule for applying the cross product is this:

$$\hat{i} \times \hat{i} = \hat{0} \quad \hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{i} = -\hat{k}$$
$$\hat{j} \times \hat{j} = \hat{0} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{j} = -\hat{i}$$
$$\hat{k} \times \hat{k} = \hat{0} \quad \hat{k} \times \hat{i} = \hat{j} \quad \hat{i} \times \hat{k} = -\hat{j}$$

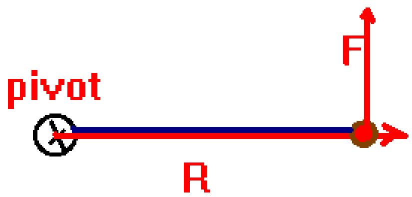
Results are positive if a positive permutation on (i,j,k) happens; negative otherwise.
Here are the positive permutations: $(\hat{i}, \hat{j}, \hat{k})(\hat{j}, \hat{k}, \hat{i})(\hat{k}, \hat{i}, \hat{j})$.

There is an animation showing the two types of permutations which is on our website.
Any cross product between the same two vectors is zero.
You can develop your own right hand rule … here is a recommendation:
(1) Place your hand at a pivot and hold your thumb, index, and middle fingers at right angles to each other.
(2) Let your index finger point in the direction that the radius vector points (away from the pivot)
(3) Let your middle finger point in the direction of the force vector.
(4) Your thumb will then point in the direction of the torque.
Sometimes it helps to write letters on your fingers for quick reference.
The magnitude of the torque can be obtained from:

$$|\vec{\Gamma}| = |\vec{R}||\vec{F}|\sin(\theta)$$

where θ is the angle between R and F. Notice that it is only the ***perpendicular component*** of F which produces a torque. You could also say it's only the perpendicular part of the distance that is important, hence the term moment arm.

(1) A force F acts on a point mass which is held by a massless rigid rod at a distance R from a pivot point and at an angle of 90°. Describe what happens.



According to Newton's law:

$$\vec{F} = m\vec{a} \;.$$

Let's take the cross product of both sides of this equation with the radius:

$$\vec{R} \times \vec{F} = m\vec{R} \times \vec{a} \Rightarrow \vec{\Gamma} = m\vec{R} \times \vec{a} \;.$$

It's probably not very clear in this format what the meaning of the right hand side of this is, but if you remember how we defined angular acceleration, (and… remember the thing will begin to rotate) then we can write this in a different way, concentrating on magnitudes:

$$|\vec{\Gamma}| = mR^2 \left|\frac{a}{R}\right| = mR^2 |\alpha|$$

The term mR^2 is particularly an important term since it seems to play the same role for rotational motion that mass played for translational motion. We call this the ***moment of inertia,*** I. With this definition, we see a nice simple connection between torque and angular acceleration:

$$\Gamma = I\alpha$$

The "correct" way to read this equation is … a torque produces an angular acceleration in the direction of the torque. But, it's also correct to read this the other way around … an angular acceleration produces a torque.

The **moment of inertia** is a term which is more generally defined by looking at mass "orbiting" about a pivot at a distance R. For discrete point masses, the moment of inertia is defined by

$$I = \sum_{\text{all masses}} m_i R_i^2$$

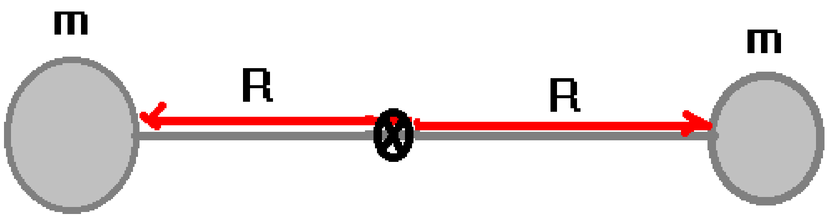
with the requirement that all the radii lie in one plane. For continuous mass distributions, the definition is similar and I refer you to the table in your text for various moments of inertia. On tests, either I'll give you moments of inertia or you'll need to be able to apply the definition for discrete masses.

For calculus students, the generalization of this to a continuous mass distribution is:

$$I = \int_{\text{all mass}} R_{\perp}\, d^3m$$

where $r_{\perp}$ is the distance from a pivot axis to a mass point. A more advanced study (than our present course) would show that the moment of inertia is actually a tensor composed of 9 elements in general and it is mathematically designated as $\overleftrightarrow{I}$. Here is a reference for moments of inertia for some solids: http://scienceworld.wolfram.com/physics/MomentofInertia.html

(2) A dumbbell consists of 2 point masses connected by a massless rod and is pivoted about its center of mass as shown below. Calculate the moment of inertia of the system about the center of mass.



Solution: The system consists of discrete point masses. Thus, we can directly apply the formula above to find the moment of inertia:

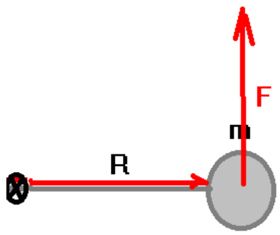
$$I = \sum_{\text{all mass}} m_i R_i^2$$

If we apply this to the present situation, we then find:

$$I = mR^2 + mR^2 = 2mR^2$$

Suppose we had 4 masses arranged at right angles. Then we would have $I=4mR^2$. You can keep dividing the masses into ever smaller increments and arranging them into a circle with the result that you will eventually get close to the moment of inertia of a ring about its center which is given by $I=MR^2$ where M is the total mass of the ring. It is also interesting to note that if you have a solid disk, the moment of inertia of the disk about its center is given by $I=\frac{1}{2}MR^2$.

(3) A point mass m is connected by a massless rod at a distance R from an infinitely strong pivot. A force F is applied to the mass at right angles. Describe what happens. The force is then removed. Describe what happens.
Solution:



Here, the torque is given by $\vec{\Gamma}=\vec{R}\times\vec{F}$. Application of the RHR shows that the torque points out of the page. This would correspond to a positive torque. The mass will undergo an angular acceleration given by

$$\Gamma=I\alpha$$

We can calculate the amount of angular acceleration by looking at the magnitude of the torque:

$$\Gamma=RF=I\alpha\Rightarrow\alpha=\frac{RF}{I}$$

We can use the rotational equations of motion to determine how the mass moves through time: $\theta=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2 : \omega=\omega_0+\alpha t : \omega^2=\omega_0^2+2\alpha(\Delta\theta)$, provided the force is constant.

Now, suppose when the mass has achieved an angular velocity ω, the torque is removed. We'll also disregard friction here.
At the instant the torque is removed, we must still have $\Gamma=I\alpha$. Let's write this in another way:

$$\Gamma=I\alpha=I\frac{\Delta\omega}{\Delta t}=\frac{\Delta(mR^2\omega)}{\Delta t}=0$$

This means that the quantity $I\omega=mR^2\omega$ won't change in time. Since $v=\omega R$, we can write this in an even more suggestive manner: $mR^2\omega=R(mv)=RP=|\vec{R}\times\vec{P}|$ which is extremely suggestive of a new quantity, the angular momentum. In the absence of external torques, the angular momentum is conserved, and the angular momentum is defined by $\vec{L}=\vec{R}\times\vec{P}$. Also note that $L=I\omega$

This provides us with another way to write the effects of a torque: $\vec{\Gamma}=\frac{\Delta\vec{L}}{\Delta t}$. It is the

change of angular momentum which is in the direction of the applied torque.

The actual direction of the angular momentum is obtained by another right hand rule which I'll show in class.

(4) A point mass m is connected by a massless rod at a distance R from an infinitely strong pivot. The mass is moving with a tangential velocity v. What is the kinetic energy of the system?
Solution:
ofthis are:
Imagine that you take a snapshot of the system. At that point, in the photo, the mass has a total kinetic energy $K=\frac{1}{2}mv^2$. Since the mass ultimately is rotating, we'll want to write this in a circular manner:

We use $v=\omega R$ in the expression for the kinetic energy to get:

$$K=\frac{1}{2}mv^2 \Rightarrow \frac{1}{2}m\left(R^2\omega^2\right)=\frac{1}{2}mR^2\omega^2=\frac{1}{2}I\omega^2$$

(5) Showing details, calculate the moment of inertia for a solid sphere of radius b with a constant volume mass density ρ and total mass M. Then calculate the moment of inertia for two concentric thin shelled cylinders of radius a and b with masses m_a and m_b about the symmetry axis. Then calculate the moment of inertia for a solid disk of radius r with a point mass m at a distance b from the center of the disk about the perpendicular axis through the center of the disk.

Solution: It is possible to calculate the moment of inertia directly for continuous distributions, with reference to a single axis. The way you do this is to form the integral:

$$I=\int_{\text{all mass}} r_\perp^2\, dm$$

We will not be doing this to any great extent in our present class but I want to show you one example. Let's calculate the moment of inertia of a sphere about a diameter: the sphere has a constant volume mass density ρ and a radius b. The distance from the diameter to any point within the sphere is given by (note that the perpendicular distance can be obtained as the magnitude of the cross product of a vector along the pivot axis and the radius vector): for example, here the pivot axis is the diameter of the sphere (the z-direction) and the radius is r so $r_\perp = r\sin(\theta)$. The element of mass is given by: $dm=\left[\frac{M}{\frac{4}{3}\pi b^3}\right][2\pi] r^2 dr \sin(\theta)\, d\theta$. We thus can calculate the moment of inertia as:

$$I=\left[\frac{M}{\frac{4}{3}\pi b^3}\right][2\pi]\int_{r=0}^{r=b} r^4 dr \int_{\theta=0}^{\theta=\pi}\sin^3(\theta)\, d\theta=\left[\frac{3M}{2b^3}\right]\frac{b^5}{5}\left[\frac{1}{12}\left(\cos(3\theta)-9\cos(\theta)\right)\right]_{\theta=0}^{\theta=\pi}$$

$$=\left[\frac{3M}{2b^3}\right]\frac{b^5}{5}\frac{1}{12}[-1+9-1+9]=\left[\frac{3M}{2b^3}\right]\frac{b^5}{5}\frac{16}{12}=\frac{2}{5}Mb^2$$

where M is the mass of the sphere. You can also obtain the moments of inertial for more complicated systems such as, for example, two concentric thin shelled cylinders of radius a and b with masses m_a and m_b about the symmetry axis. In this case, the moment of inertia would be given by: $I=I_a+I_b=M_a a^2+M_b b^2$. One final example might be the case of a solid disk of radius r with a point mass m at a distance b from the center of the disk. The moment of inertial about the perpendicular axis through the center of the disk is given by: $I=I_{disk}+mb^2=\frac{1}{2}Mr^2+mb^2$ but rotations are not stable about this axis. To show this is … well it's not the easiest thing in the world but is a standard part of an advanced course in physics.