r23	Physics 240: Worksheet 16	Name ________________

Non-uniform circular motion

Each of these problems involves non-uniform circular motion with a constant α.

(1) Obtain each of the equations of motion for non-uniform circular motion under a constant acceleration, α. Show how these equations can be modified to obtain the equations of motion when the angular acceleration varies (i) as a function of time and (ii) as a function of angular velocity.

(2) A wheel is fixed at its center and has a radius of 0.7 m. At t=0, the wheel is at =0, is not moving but is subject to an acceleration α=0.05 “rad”/s^2. Answer the following at t=10 s:
(a) What is $\omega(10)$.
(b) What is $\theta(10)$.
(c) What is the tangential velocity of a point on the rim at this time?
(d) What is the tangential acceleration of a point on the rim?
(e) What is the centripetal acceleration of a point on the rim?
(f) What is the magnitude of the total acceleration of a point on the rim?

(3) A wheel initially at rest is subject to a constant angular acceleration. After 20 seconds, the wheel has moved through an angle of 800 degrees.
(a) What is the angular acceleration of the wheel?
(b) What is the angular velocity at 20 seconds?

(4) A disk is initially rotating at 45 revolutions/minute. By placing a finger on the disk, it is observed that the disk stops in a time of 1.50 s. Answer the following:
(a) what is the average angular acceleration?
(b) what is the angle that the disk turns through during this time?

(5) A wheel initially at rest is rotated with α=3 rad/s^2 for 10 s. It is then brought to rest in 5 revolutions. Answer the following:
(a) What was the angular acceleration required?
(b) How long did it take to stop the wheel?

(1) Obtain each of the equations of motion for non-uniform circular motion under a constant acceleration, α. Show how these equations can be modified to obtain the equations of motion when the angular acceleration varies (i) as a function of time and (ii) as a function of angular velocity.
Solution: The equations describing translational motion in worksheet 2 directly apply here if you change variables in the following way:

$$x \rightarrow \theta : v \rightarrow \omega : a \rightarrow \alpha$$

α is the angular acceleration which is defined by

$$\alpha \equiv \frac{d\omega}{dt} \quad .$$

This turns out to be a vector quantity but we'll just worry about a positive or negative direction. If the object tends to increase in a counter-clockwise direction then we'll call this positive. The other direction is negative.

The equations of motion for non-uniform circular motion directly transfer from those for translational motion:

α=constant ONLY here!

Translational Equation	**Rotational Equation**
X	θ
v	ω
a	α
$x = x_0 + v_{0,x} t + \frac{1}{2} a_x t^2$	$\theta = \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2$
$v_x = v_{x,0} + a_x t$	$\omega = \omega_0 + \alpha t$
$v_x^2 = v_{0,x}^2 + 2 a_x (\Delta x)$	$\omega^2 = \omega_0^2 + 2\alpha(\Delta\theta)$

You might say that there's a big difference in that for translational motion we have three directions and there is only one (it seems) for rotational motion. This is actually not the case … we can have independent rotations about three orthogonal (=perpendicular) axes but funny things happen over long times which would be studied in more advanced courses. For our course, we'll worry about rotation about only one axis.

240: What if α is not a constant?

Now if the angular acceleration varies as a function of time as:

$$\omega = \omega_0 + \frac{1}{2} b t^2 \quad ,$$

(with b a constant) we can find the angular velocity by integration:

$$\Delta\omega = \int_{t_i}^{t_f} b t \, dt = \frac{1}{2} b \left[t_f^2 - t_i^2 \right] \quad .$$

Now if the initial time is at t=0, we obtain:

$$\omega = \omega_0 + \frac{1}{2} b t^2 \quad .$$

If the angular acceleration varies as $\alpha = b\omega$. In order to obtain the angle, we integrate again:

$$\Delta\theta = \int_{t=0}^{t} \omega_0 dt = \int_{t=0}^{t} \frac{1}{2} b t^2 dt = \omega_0 t + \frac{1}{6} b t^3 \quad .$$

If the angular acceleration varies as a function of angular velocity as:

$$\alpha = b\omega$$

(where b is constant), we need to look at the definition of angular acceleration:

$$\alpha = \frac{d\omega}{dt} \Rightarrow b\omega = \frac{d\omega}{dt} \Rightarrow \frac{d\omega}{\omega} = b\,dt$$
$$\int_{\omega_0}^{\omega} \frac{d\omega}{\omega} = b \int_{t=0}^{t} dt = bt \quad .$$

We can thus obtain:

$$<\omega> = \frac{\Delta\theta}{\Delta t} \quad .$$

We can find the angle turned through by looking at the definition of angular velocity:

$$\omega = \frac{d\theta}{dt} \Rightarrow \omega_0 e^{bt} = \frac{d\theta}{dt} \Rightarrow d\theta = \omega_0 e^{bt} dt$$
$$\Rightarrow \Delta\theta = \omega_0 \int_0^t e^{bt} dt = \frac{\omega_0}{b}\left[e^{bt} - 1\right] \quad .$$

Of course, the average angular velocity is given by:

$$<\omega> = \frac{\Delta\theta}{\Delta t}$$

which is quite different from the instantaneous angular velocity.

If we further have a **rigid body** (***which is defined by a body which has α the same throughout the body***), then there is a connection between the two accelerations and velocities which comes from the arc measurement formula. For s=arc length, the tangential velocity and the angular velocity are related by:

$$s = R\theta \Rightarrow \left(v = \frac{ds}{dt}\right) = \left(R\omega = R\frac{d\theta}{dt}\right) \Rightarrow v = \omega R$$

and the tangential acceleration and the angular acceleration are related by:

$$\left(a_t = \frac{dv_t}{dt}\right) = \left(R\frac{d\omega}{dt} = R\alpha\right) \Rightarrow a = \alpha R \quad .$$

In general, you don't necessarily have a rigid body. In order to find the vector relevant to the total acceleration, you'll need to do this vectorially:

$$\vec{a} = \vec{a}_{centripetal} + \vec{a}_{tangential} \quad ,$$

where the tangential acceleration is the change in tangential velocity with respect to time.

(2) A wheel is fixed at its center and has a radius of 0.7 m. At t=0, the wheel is at θ=0, is not moving but is subject to an acceleration α=0.05 "rad"/s^2. Answer the following at t=10 s:
(a) What is $\omega(10)$
(b) What is $\theta(10)$
(c) What is the tangential velocity of a point on the rim at this time?
(d) What is the tangential acceleration of a point on the rim?
(e) What is the centripetal acceleration of a point on the rim?
(f) What is the magnitude of the total acceleration of a point on the rim?

Solution:

(a) $\omega=\omega_0+\alpha t \Rightarrow \omega(10)=0+0.05(10)=0.5\ \frac{\text{rad}}{\text{s}}$

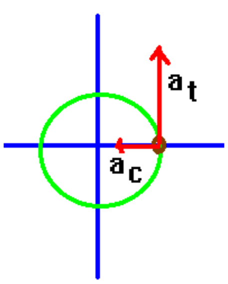
(b) $\theta=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2 \Rightarrow \theta(10)=0+0+\frac{1}{2}(0.05)(10)^2=2.5\ \text{rad}$

(c) $v=\omega R \Rightarrow v(10)=\omega(10)R=0.7(0.5)=0.35\frac{\text{m}}{\text{s}}$

(d) Since the wheel is a rigid body (I'm assuming it's not made of jelly) we can use the connection between the accelerations:

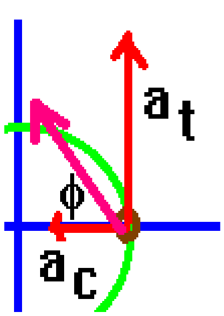
$a_t=\alpha R \Rightarrow a_t=(0.05)(0.7)=0.035\ \frac{\text{m}}{\text{s}^2}$

(e) $a_c=\frac{v^2}{R}=\frac{(0.35)^2}{0.7}=0.175\frac{\text{m}}{\text{s}^2}$



(f) Imagine you take a snapshot of the situation at the instant that the desired point is exactly along the x-axis. Then, a plot of the vectors looks like the picture below:
The two vectors are at right angles to each other. We can thus find the magnitude of the acceleration by direct application of

$|\vec{a}|=\sqrt{a_c^2+a_t^2}=\sqrt{0.175^2+0.035^2}=0.178\frac{\text{m}}{\text{s}^2}$.



We can find out the angle which the resultant acceleration is pointed in:

$\tan(\phi)=\frac{a_t}{a_c}=\frac{0.035}{0.175}=0.2 \Rightarrow \phi=\tan^{-1}(0.2)=11.3^0$.

At this instant in time, the total acceleration is pointed at an angle of 11.3^0 above the (-x) axis as is shown in the second figure below. It will always make an angle of 11.3 degrees with respect to the direction of the centripetal acceleration.

(3) A wheel initially at rest is subject to a constant angular acceleration. After 20 seconds, the wheel has moved through an angle of 800 degrees.
(a) What is the angular acceleration of the wheel?
(b) What is the angular velocity at 20 seconds?
Solution:
Probably the first problem you'll note with this is that the total angle is given in degrees but we've got to work in radians. I did this intentionally so that you can be sure you know exactly how to make this conversion correctly:

$$\theta_{radians} = \frac{2\pi}{360^0}\theta_{degrees} \Rightarrow \theta = \frac{2\pi}{360} x\, 800 = 13.96 \text{ rad} \quad .$$

Now we can answer the problem.

(a) $$\Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \Rightarrow \alpha = \frac{2\Delta\theta}{t^2} = \frac{2(13.96)}{20^2} = 0.07 \frac{\text{rad}}{\text{s}^2} \quad .$$

Notice that the problem did say "the angle moved through" which means Δθ.

(b) Now that we have the angular acceleration, finding the angular velocity is easy. I'll do it **2 ways**:

$$\omega^2 = \omega_0^2 + 2\alpha(\Delta\theta) \Rightarrow \omega = \pm\sqrt{2\alpha(\Delta\theta)} = \pm\sqrt{2(0.07)(13.96)} = 1.4 \frac{\text{rad}}{\text{s}}$$

I chose the positive solution here since the angle moved through was positive and the angular acceleration was also positive although with other initial conditions, the correct choice may have been negative.

Here is another way to get the same result:

$$\omega = \omega_0 + \alpha t \Rightarrow \omega = 0.07(20) = 1.4 \frac{\text{rad}}{\text{s}} \quad .$$

(4) A disk is initially rotating at 45 revolutions/minute. By placing a finger on the disk, it is observed that the disk stops in a time of 1.50 s. Answer the following:

(a) what is the average angular acceleration?
(b) what is the angle that the disk turns through during this time?
Solution:
You'll notice that the angular velocity is again not given in radians/second. You will need to convert to these terms to stay within the SI system. Thus:

$$45\frac{\text{revolutions}}{\text{minuts}}\text{x}\frac{1\,\text{minute}}{60\,\text{seconds}}\text{x}\frac{2\pi\,\text{radians}}{1\,\text{revolution}}=4.71\ \frac{\text{rad}}{\text{s}}=\omega_0\ .$$

(a) $$\omega=\omega_0+\alpha t\Rightarrow\omega-\omega_0=\alpha t\Rightarrow\alpha=\frac{\omega-\omega_0}{t}=\frac{0-4.71}{1.50}=-3.14\ \frac{\text{rad}}{\text{s}^2}$$

(b) Now that we have the angular acceleration, we can find the angle in different ways. I'll do it **2 ways**:

$$\omega^2=\omega_0^2+2\alpha(\Delta\theta)\Rightarrow\Delta\theta=\frac{\omega^2-\omega_0^2}{2\alpha}=-\frac{4.71^2}{2(-3.14)}=3.53\text{ rad}$$

$$\Delta\theta=\omega_0 t+\frac{1}{2}\alpha t^2=(4.71)(1.5)+\frac{1}{2}(-3.14)(1.5^2)=7.065-3.542=3.53\text{ rad}$$

(5) A wheel initially at rest is rotated with $\alpha=3$ rad/s^2 for 10 s. It is then brought to rest in 5 revolutions. Answer the following:
(a) What was the angular acceleration required to stop the wheel?
(b) How long did it take to stop the wheel?

It is important in this problem to distinguish between the two angular accelerations!

Solution: We need to find first how fast it was moving at the end of the initial 10s . I will call this first angular acceleration α_1 . The angular velocity after 10 s is given by:

$$\omega=\omega_0+\alpha_1 t \Rightarrow \omega=3(10)=30\ \frac{\text{rad}}{\text{s}} \quad ,$$

where the subscript 1 indicates the first acceleration.

We can now answer everything else but first, we ought to convert 5 revolutions to radians …

$$5\ \text{revolutions}\ \text{x}\ \frac{2\pi\ \text{radians}}{1\ \text{revolution}}=31.4\ \text{radians} \quad .$$

The second angular acceleration I will call α_2 .

(a) $\omega^2=\omega_0^2+2\alpha_2(\Delta\theta)\Rightarrow\alpha_2=\frac{\omega^2-\omega_0^2}{2(\Delta\theta)}=-\frac{30^2}{2(31.4)}=-14.33\ \frac{\text{rad}}{\text{s}^2}$

(b) $\omega=\omega_0+\alpha_2 t\Rightarrow t=\frac{\omega-\omega_0}{\alpha_2}=\frac{-30}{-14.33}=2.09\,\text{s}$