r20	Physics 240: Worksheet 17	Name: _______________

This worksheet uses the concepts of rotational kinetic energy and angular momentum.

(1) A point mass m is connected by a massless rod at a distance R from an infinitely strong pivot. The mass is moving with a tangential velocity v. What is the kinetic energy of the system?

(2) A solid wheel is rolled along a horizontal surface at a constant velocity v without slipping. Find the total kinetic energy of the system.

(3) A solid sphere and a hollow sphere are rolled down an inclined plane, starting from rest, through a height h. Which one gets to the bottom first?

(4) Suppose a star of radius R_1 has a period of 20 days. The star suddenly collapses to a radius R_2 which is smaller by a factor of 10000 without losing mass. What is the new period of the star? You may assume that the moment of inertia of the star is that of a solid sphere $I=\frac{2}{5}mR^2$. Also, compare the ratio of Kinetic energy before and after the collapse.

(5) Explain the precession of a bicycle wheel as shown in class.

r20	Physics 240: Worksheet 17	Name: _______________

(1) A point mass m is connected by a massless rod at a distance R from an infinitely strong pivot. The mass is moving with a tangential velocity v. What is the kinetic energy of the system?

Solution:
You'll want to refer to one of the animated gifs for further help here.
http://logcabinphysics.x10.bz/animations/rotationalke1.gif

The details of this are:
Imagine that you take a snapshot of the system. At that point, in the photo, the mass has a total kinetic energy $K=\frac{1}{2}mv^2$. Since the mass ultimately is rotating, we'll want to write this in a circular manner:

We use $v=\omega R$ in the expression for the kinetic energy to get:

$$K=\frac{1}{2}mv^2 \Rightarrow K=\frac{1}{2}m\left(R^2\omega^2\right)=\frac{1}{2}mR^2\omega^2=\frac{1}{2}I\omega^2$$

(2) A solid wheel is rolled along a horizontal surface at a constant velocity v without slipping. Find the total kinetic energy of the system.

Solution: The moment of inertial for a solid wheel is $I=MR^2$. Now, if the wheel is rotating about its axis, the rotational kinetic energy is given by: $K_{rot}=\frac{1}{2}I\omega^2$. If, in addition, the center of mass of the wheel is translating, then the translational kinetic energy is given by $K_{trans}=\frac{1}{2}mv^2$. To find the total kinetic energy of the system, add the individual contributions to find:

$$K_{total}=K_{rot}+K_{trans}=\frac{1}{2}I\omega^2+\frac{1}{2}mv^2$$

If that's all we knew about the problem, we'd have to stop here. In this problem, however, we are additionally told the wheel rolls without slipping. This provides a connection between ω and v which is as follows:

$$s=R\theta \Rightarrow v=\omega R \Rightarrow \omega=\frac{v}{R} \quad .$$

Here is an illustration of rolling without slip:
http://logcabinphysics.x10.bz/simulationmovies/RollWithoutSlip/RollWithoutSlip01.wmv

We use this in the total kinetic energy to find:

$$K_{total}=\frac{1}{2}I\omega^2+\frac{1}{2}mv^2=\frac{1}{2}mR^2\left(\frac{v}{R}\right)^2+\frac{1}{2}mv^2=\frac{1}{2}mv^2(1+1)=mv^2 \quad .$$

r20	Physics 240: Worksheet 17	Name: _______________

Suppose that a round body had, more generally, a moment of inertia given by

$$I=\beta m R^2$$

and the condition for rolling without slipping was imposed. Then more generally, the total kinetic energy would be given by:

$$K_{total}=\frac{1}{2} m v^2 (\beta+1)$$

It's interesting to note that when a wheel rolls without slip, there is one point in contact with the surface which is stationary meaning the frictional force which keeps the wheel rotating is the static frictional force.
The condition for rolling with slip is more complicated and is best solved by application of Newton's laws directly to the system.

(3) A solid sphere and a hollow sphere of the same mass are rolled down an inclined plane, starting from rest, through a height h. Which one gets to the bottom first?

Solution:

Looking back at the results from problem 2, we had the total kinetic energy given by: $K_{total}=\frac{1}{2}mv^2(\beta+1)$. Here, we are comparing the results for two different objects. According to the conservation of total mechanical energy, we have for each of the objects:

$$\Delta U+\Delta K=0:\Delta U=0-mgh:\Delta K=K_{final}-0$$
$$\Rightarrow -mgh+K_{final}=0$$

Using the expression above and writing this twice, we find:

$$mgh=\frac{1}{2}mv_1^2(\beta_1+1)$$
$$mgh=\frac{1}{2}mv_s^2(\beta_2+1)$$

We can now solve this for the ratio of the velocities (and, we know that the fastest one wins the race) ....

$$mgh=\frac{1}{2}mv_1^2(\beta_1+1)\Rightarrow v_1^2=\frac{2gh}{\beta_1+1}$$
$$mgh=\frac{1}{2}mv_2^2(\beta_2+1)\Rightarrow v_2^2=\frac{2gh}{\beta_2+1}$$

Thus:

$$\frac{v_1^2}{v_2^2}=\frac{\frac{2gh}{\beta_1+1}}{\frac{2gh}{\beta_2+1}}=\frac{\beta_2+1}{\beta_1+1}\Rightarrow\frac{v_1}{v_2}=\sqrt{\frac{\beta_2+1}{\beta_1+1}}$$

Let's see how this works for the present system:
Here, let's let the solid sphere be 1 and the hollow sphere be 2. The moments of inertia are given by:

$$I_{solid}=\frac{2}{5}mR^2\Rightarrow\beta_1=\frac{2}{5}$$
$$I_{hollow}=I_2=\frac{2}{3}mR^2\Rightarrow\beta_2=\frac{2}{3}$$

We thus find the ratio of velocities as:

$$\frac{v_1}{v_2}=\sqrt{\frac{\beta_2+1}{\beta_1+1}}=\sqrt{\frac{\frac{2}{3}+1}{\frac{2}{5}+1}}=\sqrt{\frac{\frac{5}{3}}{\frac{7}{5}}}=\sqrt{\frac{5}{3}x\frac{5}{7}}-\sqrt{\frac{25}{21}}=\sqrt{1.19}=1.09$$

Since $v_1=1.09v_2$, the sphere corresponding to "1" gets to the bottom first, namely the solid sphere. You can develop an argument for this which goes as follows: an amount of energy is divided (unequally) between rotational and translational terms. The less energy that goes into rotational motion, the more that there is for translational motion. You'll also note in this formulation, this is independent of mass although mass does play a role when one considers friction (indents the track more or less) in a more advanced treatment.

(4) Suppose a star of radius R_1 has a period of 20 days. The star suddenly collapses to a radius R_2 which is smaller by a factor of 10000 without losing mass. What is the new period of the star? You may assume that the moment of inertia of the star is that of a solid sphere $I=\frac{2}{5}mR^2$. Also, compare the ratio of Kinetic energy before and after the collapse.

$$\vec{\Gamma}=I\vec{\alpha}=I\frac{d\vec{\omega}}{dt} \quad .$$

Now let's look at what happens when the torque is removed.

$$\vec{0}=\frac{d(I\vec{\omega})}{dt}\Rightarrow I\omega=\text{constant}$$

For a point mass, we have:

$$\vec{L}=I\vec{\omega}$$

L is called the angular momentum, and it is conserved in the absence of external torques.

For systems which are rotating about one principle (symmetry) axis, the direction of the angular momentum is in the same direction as the angular velocity. The direction of the angular velocity is found by what I call the right hand rule number 2. I've made a simulation of how you find this direction.

http://logcabinphysics.x10.bz/simulationmovies/RHR-1/RHR1-003.wmv

Essentially, you grip the axis of rotation with the palm of your hand with your fingers pointing in the direction of the tangential velocity. Your thumb then points in the direction of the angular velocity and thus the angular momentum. In the absence of external torques, angular momentum is conserved.

$$\vec{L}_{before}=\vec{L}_{after} \quad .$$

The **angular momentum** is given by

$$L=I\omega=I(2\pi f)=\frac{2\pi I}{T}$$

where T is the period of rotation. We can use this together with the moment of inertia to find that

$$L=\frac{2\pi\left(\frac{2}{5}mR^2\right)}{T} \quad .$$

The direction of the angular momentum is the same before as after. Let's find the new period:

$$\frac{2\pi\left(\frac{2}{5}mR_1^2\right)}{T_1}=\frac{2\pi\left(\frac{2}{5}mR_2^2\right)}{T_2}\Rightarrow\frac{R_1^2}{T_1}=\frac{R_2^2}{T_2}\Rightarrow T_2=T_1\left(\frac{R_2}{R_1}\right)^2$$

In the present case, we can find the new period then from

$$T_2=T_1\left(\frac{R_2}{R_1}\right)^2\Rightarrow T_2=20\left(\frac{R_2}{1\times10^4R_2}\right)^2=\frac{20}{1\times10^8}=2\times10^{-7}\text{days}=0.0173\text{ s} \quad .$$

If there were a source of EM waves (a sunspot perhaps) on the star, they would be detected with a frequency of f=57.9 Hz. We can also find the kinetic energy before and after:

$$K_{before}=\frac{1}{2}I_{before}\omega_{before}^2 \quad \text{and } K_{after}=\frac{1}{2}I_{after}\omega_{after}^2$$

$$\frac{K_{before}}{K_{after}}=\frac{\frac{1}{2}I_{before}\omega_{before}^2}{\frac{1}{2}I_{after}\omega_{after}^2}=\frac{R_{before}^2\left(\frac{2\pi}{T_{before}}\right)^2}{R_{after}^2\left(\frac{2\pi}{T_{after}}\right)^2}=\left(\frac{R_{before}T_{after}}{R_{after}T_{before}}\right)^2=\left(\frac{1\times10^4}{1}\times\frac{2\times10^{-7}}{20}\right)^2=1\times10^{-8}$$

or

$$K_{after}=1\times10^8K_{before} \quad .$$

Thus the kinetic energy of the star did increase. Sometimes text books are a little bit shy about this particular point because it can be hard to trace down the thing that supplied the energy. Here, the increase in kinetic energy came from a decrease in the gravitational potential energy.

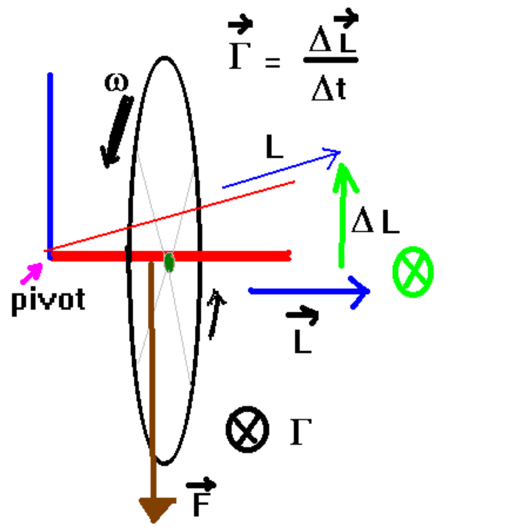
(5) Explain the precession of a bicycle wheel as shown in class.

I have provided an animation of the bicycle wheel which shows the phenomenology of the precession.
http://logcabinphysics.x10.bz/simulationmovies/BicycleWheelPrecession/PrecessionOfBicycleWheel03.wmv

The important point to remember here is that the change in direction of the angular momentum is in the direction of the external torque which is applied.

$$\text{Calculus}: \vec{\Gamma} = I\vec{\alpha} = I\frac{d\vec{\omega}}{dt} \qquad \text{Non-calculus}: \vec{\Gamma} = \frac{\Delta\vec{L}}{\Delta t}$$

Using RHR#1, the gravitational force acts upon the center of mass of the wheel which is pivoted by one end of the axel. Using RHR#1 shows that, for this orientation, the direction of the torque is into the screen (shown by the black circle). The change in angular momentum is in the same direction which thus is also into the screen. The wheel thus shows precession about the pivot in the direction shown by the green circle. If you held the end of the other axel, the precession would be in the opposite direction.



Direction of ΔL is in the direction of Γ

You can determine the direction of the torque with another right hand rule:
http://logcabinphysics.x10.bz/simulationmovies/RightHandRuleNumberTwo/RHR2-001.wmv

I will note in passing that the angular momentum is also defined by a cross product:

$$\vec{L} = \vec{R} \times \vec{P}$$

so that if you have a mass traveling with a translational momentum P at some radius R from a point, the angular momentum about that point is given as shown by the cross product above. The same simulation as before applies here also.
http://logcabinphysics.x10.bz/simulationmovies/RightHandRuleNumberTwo/RightHandRuleNumberTwo01.wmv