Special Calculus notes related to centripetal acceleration

A point moving in uniform circular motion with angular velocity ω at a time t and at a distance b from the origin is described by a position vector given by:

$$\vec{R} = b\cos(\omega t)\hat{x} + b\sin(\omega t)\hat{y}$$

The velocity at each instant in time is given by the derivative of the position vector:

$$\vec{v} = \frac{d\vec{R}}{dt} = -\omega b\sin(\omega t)\hat{x} + \omega b\cos(\omega t)\hat{y} \equiv \vec{v}_t$$

Note that the magnitude of the tangential velocity is simply related to ω:

$$|\vec{v}_t| = \omega b$$

which, for a particle traveling at a radius r from the center of a circle reduces to the familiar form:

$$v_t = \omega r \quad .$$

The velocity here is tangent to the curve (since b is a constant) and it is the velocity that determines what the curve ultimately looks like. The vector pointing in the direction of the velocity vector is given by the unit vector pointing in the direction of the velocity (this is called the tangent vector). The unit tangent vector is easy to calculate with the rules of how to calculate unit vectors:

$$\hat{T} = \frac{\vec{v}_t}{|\vec{v}_t|} = \frac{-\omega b\sin(\omega t)\hat{x} + \omega b\cos(\omega t)\hat{y}}{\sqrt{\vec{v}_t \bullet \vec{v}_t}} = \frac{-\omega b\sin(\omega t)\hat{x} + \omega b\cos(\omega t)\hat{y}}{\omega b}$$

$$\Rightarrow \hat{T} = -\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}\,.$$

Now you would like to say that this vector points in the angular direction, and you indeed can say that simply by defining the unit vector in the angular direction:

$$\hat{\theta} = -\sin(\omega t)\hat{x} + \cos(\omega t)\hat{y}\,.$$

You would also like to know that the position vector in the radial direction is also given by following the rules for calculation of unit vectors:

$$\hat{R} = \frac{\vec{R}}{|\vec{R}|} = \frac{b\cos(\omega t)\hat{x} + b\sin(\omega t)\hat{y}}{b} = \cos(\omega t)\hat{x} + \sin(\omega t)\hat{y}.$$

You can also easily prove that these two vectors are perpendicular to each other:

$$\hat{R} \bullet \hat{\theta} = -\sin(\omega t)\cos(\omega t) + \cos(\omega t)\sin(\omega t) = 0$$

An additional detail: you might want to replace the argument by $\omega t \equiv \theta$ if you’re not dealing with problems involving motion.

When the point is moving around a circle, the velocity is given by:

$$\vec{v} = |\vec{v}|\hat{T} = |\vec{v}|\hat{\theta}$$

where the last equality results only in this case for a particle moving on a circular path.

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We can now find the acceleration. As usual the acceleration is given by the time derivative of the velocity:

$$\vec{a}=\frac{d\vec{v}}{dt}=d\frac{(|\vec{v}|\hat{\theta})}{dt}=\hat{\theta}\frac{d|\vec{v}|}{dt}+|\vec{v}|\frac{d\hat{\theta}}{dt}$$

Note that you have be extremely careful with how you do this so that you do not neglect the time dependence of the unit vector in the angular direction! In fact, for uniform circular motion, that is where all the excitement is.

If the speed of the particle is constant (as is required for uniform circular motion), we then have:

$$\vec{a}=|\vec{v}|\frac{d\hat{\theta}}{dt}=|\vec{v}|\frac{d[-\sin(\omega t)\hat{x}+\cos(\omega t)\hat{y}]}{dt}=|\vec{v}|[-\omega\cos(\omega t)\hat{x}-\omega\sin(\omega t)\hat{y}]=-\omega|\vec{v}|\hat{R}$$

This acceleration is the "centripetal" acceleration, and is of magnitude:

$$a_c=\frac{v_t^2}{r}=\omega^2 r$$

and the negative sign says that it points towards the center of the circle.
We can also obtain the angular acceleration. However since this is uniform circular motion, the angular acceleration would be zero. Also the vector operation needed to find the angular velocity vector and the angular acceleration vector is slightly different.

In calculus III, the centripetal acceleration may be referred to as the normal component of the acceleration, in physics we call this the centripetal acceleration. Thus: if v is the speed of the particle, we then have:

$$|\vec{a}_c|=\omega v_t=\omega^2 r=\frac{v_t^2}{r.}$$

Now for a particle undergoing circular motion, it will have a connection between the distance it goes through and its velocity. The distance in one revolution at a radius b from the origin is given by:

$$s=\int ds=\int_{\theta=0}^{\theta=2\pi} b\,d\theta=2\pi b.$$

If T is the time for one revolution, then in this time T, we would require:

$$v_t T=2\pi b\Rightarrow v_t=\frac{2\pi b}{T}=2\pi f b=\omega b$$

where f is the frequency in Hz, and ω is the angular frequency in "rad"/s.
Thus the centripetal acceleration is given by:

$$a_c=\omega v_t=\omega^2 b=\frac{v_t^2}{b}.$$

$$a_c=\omega v=\omega^2 b$$

If your circle has a radius R instead of b, we then have:

$$a_c=\omega v_t=\omega^2 R=\frac{v_t^2}{R}.$$