

Total mechanical energy:

1/2 of the time average energy is kinetic, 1/2 is potential

$$\langle E \rangle = \langle K \rangle + \langle U \rangle$$

For a spring mass system:

$$x(t) = A \cos(\omega t + \phi); v(t) = -\omega A \sin(\omega t + \phi); \omega = \sqrt{\frac{k}{m}}$$

If the spring is at a maximum displacement, the total energy is $E = \frac{1}{2} k A^2$.

Calculate the time average kinetic energy and the time average potential energy:

$$\langle K \rangle = \frac{1}{2} m \langle v^2 \rangle = \frac{1}{2} m \omega^2 A^2 \langle \sin^2(\omega t + \phi) \rangle = \frac{1}{2} m \omega^2 A^2 \left(\frac{1}{2} \right)$$

$$m \omega^2 = m \left(\sqrt{\frac{k}{m}} \right)^2 = m \frac{k}{m} = k \Rightarrow \langle K \rangle = \frac{1}{2} k A^2 \left(\frac{1}{2} \right)$$

$$\langle U \rangle = \frac{1}{2} k \langle x^2 \rangle = \frac{1}{2} k A^2 \langle \cos^2(\omega t + \phi) \rangle = \frac{1}{2} k A^2 \left(\frac{1}{2} \right)$$

$$\langle K \rangle + \langle U \rangle = \frac{1}{2} k A^2 \left(\frac{1}{2} \right) + \frac{1}{2} k A^2 \left(\frac{1}{2} \right) = \frac{1}{2} k A^2 = \langle E \rangle = \frac{1}{2} k A^2$$

Note: to calculate the time average of cos or sin:

$$1 = \cos^2(\theta) + \sin^2(\theta); \langle 1 \rangle = \langle \cos^2(\theta) \rangle + \langle \sin^2(\theta) \rangle$$

cosine and sine behave exactly the same except displaced by a phase. This means each contributes equally to the value of 1. Thus:

$$\langle \cos^2(\theta) \rangle = \langle \sin^2(\theta) \rangle = \frac{1}{2}$$