

Monochromatic plane waves

$$\vec{E}(z, t) = \vec{E}_0 e^{i(kz - \omega t)}; \vec{B}(z, t) = \vec{B}_0 e^{i(kz - \omega t)}$$

describes waves traveling along the z-axis. Notice that the spatial and temporal dependencies are solely contained in the exponentials. The (complex) amplitudes are constant vectors here.

To get the actual fields, you would take the real parts of these waves.

Right now, these are very general and do not necessarily obey Maxwell's equations. In order to describe EM phenomena, we need to look at the restrictions which Maxwell's equations force.

Since we are in a charge-free region of space we require:

$$(\vec{\nabla} \cdot \vec{E}_0)_z = (\vec{\nabla} \cdot \vec{B}_0)_z = 0$$

In words, the disturbance amplitudes are transverse; they are perpendicular to the direction of wave propagation.

Faraday's law imposes a requirement that connects the amplitudes of the fields:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

In particular, to take the curl :

$$\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & 0 \end{bmatrix} = \frac{-\partial E_y}{\partial z} \hat{x} + \frac{\partial E_x}{\partial z} \hat{y} + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \hat{z}$$

Which gives:

$$-k(\vec{E}_0)_y = \omega(\vec{B}_0)_x; k(\vec{E}_0)_x = \omega(\vec{B}_0)_y$$

This means that if E is in the x-direction, B is in the y-direction, etc. The wave amplitudes must be perpendicular in the transverse plane. More compactly:

$$(\vec{B}_0) = \frac{k}{\omega} (\hat{z} \times \vec{E}_0)$$

This gives us the first connection: for the real amplitudes:

$$\frac{E_0}{B_0} = \frac{\omega}{k} = c$$

Which is remembered by "Every Body Claps".

Let's define the "polarization vector" as that vector pointing along the direction of the electric field. We can generalize away from the z-direction by assigning the wave number as a vector to give:

$$\vec{E} = \tilde{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \hat{n}; \vec{B} = \frac{1}{c} \tilde{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} (\hat{k} \times \hat{n}) = \frac{1}{c} \vec{k} \times \vec{E}$$

In terms of the real amplitudes, these look like:

$$\vec{E} = E_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) \hat{n}; \vec{B} = \frac{1}{c} E_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) (\hat{k} \times \hat{n})$$

Here, \hat{n} is the polarization vector (not! the normal vector) and it must obey:

$$\hat{n} \cdot \hat{k} = 0$$

because the waves are transverse.

Energy and Momentum in transverse EM (called TEM) waves.

We know how to calculate the energy density in a TEM wave:

$$u_{EM} = \frac{1}{2} \left(\epsilon_0 \vec{E} \cdot \vec{E} + \frac{1}{\mu_0} \vec{B} \cdot \vec{B} \right)$$

It is here most transparent to work with the real fields above:

$$u_{EM} = \frac{1}{2} \left(\epsilon_0 E_0^2 + \frac{1}{\mu_0} B_0^2 \right) \cos^2(\vec{k} \cdot \vec{r} - \omega t + \delta)$$

Since the amplitudes are connected by: $B_0 = \frac{1}{c} E_0 = \sqrt{\epsilon_0 \mu_0} E_0 \Rightarrow B_0^2 = \epsilon_0 \mu_0 E_0^2$

The electric energy density and the magnetic energy density are equal and the density is given by:

$$u_{EM} = \epsilon_0 E_0^2 \cos^2(\vec{k} \cdot \vec{r} - \omega t + \delta)$$

We can use the Poynting vector to determine the energy flux density for the waves propagating in the z-direction:

$$\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B}) = c \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \hat{z} = c u_{EM} \hat{z}$$

The momentum carried by an EM wave is also directly related to the Poynting vector. The momentum density is given by:

$$\vec{p}_{EM} = \frac{1}{c^2} \vec{S}$$

So for the monochromatic TEM waves in the z-direction, we have:

$$\vec{p}_{EM} = \frac{1}{c} \epsilon_0 E_0^2 \cos^2(kz - \omega t + \delta) \hat{z} = \frac{u_{EM}}{c} \hat{z}$$

We are normally only interested in the time average of these quantities, which gives (using the argument that we have often used)

$$\langle u_{EM} \rangle = \frac{1}{2} \epsilon_0 E_0^2$$

$$\langle \vec{S} \rangle = \frac{c u_{EM}}{2} \hat{z} = \frac{1}{2} c \epsilon_0 E_0^2 \hat{z}$$

$$\langle \vec{p} \rangle = \frac{u_{EM}}{2c} \hat{z} = \frac{\epsilon_0}{2c} E_0^2 \hat{z}$$

It is this time average Poynting vector dotted into a normal to a detector area that we call the Intensity (I).

We can directly relate the intensity to radiation pressure as follows for a perfect absorber:

In a time t , we have:

$$\Delta \vec{P} = a \langle \vec{p} \rangle A (c \Delta t) \Rightarrow \text{Pressure} = \frac{\Delta P}{A \Delta t} = \frac{1}{2} \epsilon_0 E_0^2 = \frac{I}{c}$$

For a perfect reflector, the pressure is twice this since the EM wave must change direction.