7.2.3 Inductance

consider two loops of wire which are spatially separated but at rest with respect to each other. In the first look, supply a current I1 which will result in a magnetic field from the first loop B1. We calculate the magnetic field from Biot-Savart:

$$\vec{B}_1=\frac{\mu_0}{4\pi}I_1\oint\frac{d\vec{L}_1 \times \hat{r}_{ip}}{r_{ip}^2}$$

The flux passing through the second loop is:

$$\Phi_2=\oiint\vec{B}_1\cdot d\vec{a}_2$$

The flux passing through loop 2 is proportional to the current in loop 1. We define the mutual inductance of the two loops based upon this:

$$\Phi_2=M_{21}I_1$$

where M_{21} is called the “mutual inductance” of the loops.

There is a standard way that the mutual inductance can be expressed:

$$\Phi_2=\oiint\vec{B}_1\cdot d\vec{a}_2=\oiint\left(\vec{\nabla}\times\vec{A}_1\right)\cdot d\vec{a}_2=\oint\vec{A}_1\cdot d\vec{L}_2$$

But the vector potential is given by:

$$\vec{A}_1=\frac{\mu_0}{4\pi}I_1\oint\frac{d\vec{L}_1}{r_{ip}}$$

So we have:

$$\Phi_2=\frac{\mu_0}{4\pi}I_1\oint\left(\oint\frac{d\vec{L}_1}{r_{ip}}\right)\cdot d\vec{L}_2$$

So we have the mutual inductance:

$$M_{21}=\frac{\mu_0}{4\pi}\oint\oint\frac{d\vec{L}_1\cdot d\vec{L}_2}{r_{ip}}$$

Your author refers to this as the Neumann formula and from this it is clear that the mutual inductance depends upon the shapes and positions of the two loops and $M_{21}=M_{12}$.

Example 7.10 A short solenoid with radius a and n1 turns per unit length is inside a much longer solenoid with n2 turns per unit length. If the inner one carries a current I1, what is the mutual inductance of the system?

If you do this problem from this viewpoint, it is hard. Instead we apply a current I2 to the long solenoid and calculate the mutual inductance that way.

The magnetic field inside the second solenoid is given by: $B=\mu_0 n2 I_2$ so the flux through the short solenoid is given by:

$$\Phi_M=(n_1 h_1)(\mu_0 n_2 I_2)(\pi a^2)\Rightarrow M_{12}=M_{21}=\mu_0 \pi a^2 n_1 n_2 h$$

If we now allow the current to vary by Faraday's law we then have:

$$emf_2=\frac{-d\Phi_2}{dt}=-M\frac{dI_1}{dt}$$

Now this changing current also introduces a emf in the circuit itself by:

$$\Phi=LI\Rightarrow emf=-L\frac{dI}{dt}$$

where L is the “self inductance”.

Example 7.11: find the self-inductance of a torrid with a rectangular cross section (inner radius a, outer radius b, height h, N turns).

From Ampere's law, the magnetic field in the torrid is:

$$B=\frac{\mu_0 NI}{2\pi s}$$

The total flux through the torrid is then:

$$\Phi_M=N\oiint \vec{B}\cdot d\vec{a}=\frac{\mu_0 N^2 I}{2\pi}h\int_a^b \frac{ds}{s}=\frac{\mu_0 N^2 I h}{2\pi}\ln\left(\frac{b}{a}\right)\Rightarrow L=\frac{\mu_0 N^2 h}{2\pi}\ln\left(\frac{b}{a}\right)$$

7.2.4 Energy in magnetic fields

We start by calculating the work required to establish a current in an inductance:

$$\text{Power}=\frac{dU_M}{dt}=-I\mathcal{E}=LI\frac{dI}{dt}\Rightarrow U_m=\frac{1}{2}LI^2$$

This is the stored magnetostatic energy when a current I is established.
We can obtain the magnetostatic energy density by this:

$$\Phi=\oiint\vec{B}\cdot d\vec{a}=\oiint\left(\vec{\nabla}x\vec{A}\right)\cdot d\vec{a}=\oint\vec{A}\cdot d\vec{l}\Rightarrow LI=\oint\vec{A}\cdot d\vec{l}$$

The magnetostatic energy is then:

$$U_M=\frac{1}{2}I\oint\vec{A}\cdot d\vec{l}=\frac{1}{2}\oint\left(\vec{A}\cdot\vec{I}\right)dl=\frac{1}{2}\oiiint\left(\vec{A}\cdot\vec{J}\right)d^3r=\frac{1}{2\mu_0}\oiiint\vec{A}\cdot\left(\vec{\nabla}x\vec{B}\right)d^3r$$

$$\vec{\nabla}\cdot\left(\vec{A}x\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}x\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}x\vec{B}\right)\Rightarrow\vec{A}\cdot\left(\vec{\nabla}x\vec{B}\right)=\vec{B}\cdot\vec{B}-\vec{\nabla}\cdot\left(\vec{A}x\vec{B}\right)$$

Now integration gives:

$$U_M=\frac{1}{2\mu_0}\left[\oiiint B^2d^3r=\oiint\left(\vec{A}x\vec{B}\right)\cdot d\vec{a}\right]$$

$$\text{because}\quad\oiiint\vec{\nabla}\cdot\left(\vec{A}x\vec{B}\right)d^3r=\oiint\left(\vec{A}x\vec{B}\right)\cdot d\vec{a}$$

What is now normally argued is to let the surface go to infinity where B will vanish so that the magnetostatic energy is expressed as:

$$U_M=\frac{1}{2\mu_0}\oiiint_{\text{all space}}B^2d^3r\quad\text{which is similar in concept to}\quad U_E=\frac{1}{2}\epsilon_0\oiiint E^2d^3r\quad.$$

We thus identify the magnetostatic energy density as:

$$u_M=\frac{B^2}{2\mu_0}\quad\text{which is again similar in concept to the electrostatic form:}\quad u_E=\frac{1}{2}\epsilon_0E^2\quad.$$

Example 7.13 Long coax cable (inner radius a, outer radius b, length h) carries I inside and -I outside. Find the magnetostatic energy.

$$B=\frac{\mu_0I}{2\pi s}\Rightarrow u_M=\frac{\left(\frac{\mu_0I}{2\pi}\right)^2}{2\mu_0}\left(2\pi h\right)\int_{s=a}^{s=b}\frac{sds}{s^2}=\frac{\left(\mu_0I^2\right)}{4\pi}h\ln\left(\frac{b}{a}\right)$$

Your author points out, and correctly so , that we could now calculate the inductance of the cable by:

$$U=\frac{1}{2}LI^2\Rightarrow L=\frac{\mu_0h}{2\pi}\ln\left(\frac{b}{a}\right)\quad.$$