Magnetic Vector Potential 2016

We can define A as that vector function such that:

$$\vec{B} \equiv \vec{\nabla} \times \vec{A}$$

Look at the curl of B:

$$\vec{\nabla} \times \vec{B} = \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

We are free to add to A any vector function whose curl vanishes. This is used to give:

$$\vec{\nabla} \cdot \vec{A} = 0$$

(a divergence does not curl and a curl does not diverge)
Then we can find A by solving:

$$\nabla^2 \vec{A} = -\mu_0 \vec{J}$$

Provided that J goes to zero at infinity, we then have:

$$\vec{A}(\vec{r}_p) = \frac{\mu_0}{4\pi} \int_{\text{all currents}} \frac{\vec{J}(\vec{r}_i)}{|\vec{r}_{ip}|} d\tau_i$$

modifications of this are:

line currents:

$$\vec{A}(\vec{r}_p) = \frac{\mu_0}{4\pi} \int_{\text{all currents}} \frac{\vec{I}(\vec{r}_i)}{|\vec{r}_{ip}|} dL_i$$

surface currents

$$\vec{A}(\vec{r}_p) = \frac{\mu_0}{4\pi} \int_{\text{all currents}} \frac{\vec{K}(\vec{r}_i)}{|\vec{r}_{ip}|} dA_i$$

Let's look at A: Since

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

Let's use the "Curl Theorem"

$$\oiint (\vec{\nabla} \times \vec{A}) \cdot d\vec{A} = \oint \vec{A} \cdot d\vec{L}$$

So in a sense, you can calculate A by calculating the magnetic flux through a surface: $\Phi_M = \oint \vec{A} \cdot d\vec{L}$ since $\oiint (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oiint \vec{B} \cdot d\vec{a} = \Phi_M$

Note I am using a to represent the area here.

This looks a lot like Ampere's law but it's actually not. This is useful for situations where the current does not vanish at infinity.

An application of this: an **ideal** solenoid of turn density n. Choose a circular path of radius s centered on the solenoid which extends to infinity. The magnetic field is uniform inside the solenoid and zero outside. Then:

$$\Phi_M = \oint \vec{A} \cdot d\vec{L} \Rightarrow A(2\pi s) = B(\pi a^2) \Rightarrow A = \frac{Ba^2}{2s}$$

If, however, our loop is inside the solenoid, we have:

$$\Phi_m = \oint \vec{A} \cdot d\vec{L} \Rightarrow A(2\pi s) = B(\pi s^2) \Rightarrow A = \frac{Bs}{2}$$

Now the vector direction needs to be such that the curl of A gives B. If the vector direction of A is in the ϕ direction, then we can check this … the hard way is to write the unit vector in Cartesian coordinates. I prefer the easier way:

In cylindrical coordinates:

$$\vec{\nabla}\times\vec{V}=\left(\frac{1}{s}\frac{\partial V_z}{\partial\phi}-\frac{\partial V_\phi}{\partial z}\right)\hat{s}+\left(\frac{\partial V_s}{\partial z}-\frac{\partial V_z}{\partial s}\right)\hat{\phi}+\frac{1}{s}\left[\frac{\partial(s V_\phi)}{\partial s}-\frac{\partial V_s}{\partial\phi}\right]\hat{z}$$

Now if the vector only has an angular component, then:

$$\vec{\nabla}\times\vec{V}=\frac{1}{s}\left[\frac{\partial(s V_\phi)}{\partial s}\right]\hat{z}-\left(\frac{\partial V_\phi}{\partial z}\right)\hat{s}$$

If the vector has no functional z-dependence then:

$$\vec{\nabla}\times\vec{V}=\frac{1}{s}\left[\frac{\partial(s V_\phi)}{\partial s}\right]\hat{z}$$

So for A, if we have that

$$\vec{A}=\frac{Ba^2}{2s}\hat{\phi}$$

then the field does vanish outside the solenoid. If:

$$\vec{A}=\frac{Bs}{2}\hat{\phi}$$

Then:

$$\frac{1}{s}\frac{\partial\left[\frac{Bs^2}{2}\right]}{\partial s}=\frac{1}{s}[Bs]=B$$

so:

$$\vec{B}=B\hat{z}$$

Thus: Inside: $\vec{A}=\frac{Bs}{2}\hat{\phi}$ and outside: $\vec{A}=\frac{Ba^2}{2s}\hat{\phi}$

Probably it's important to realize that B needs to curl around the direction of A in making the vector arguments. It almost would also seem that A needs to curl around B but this can not be completely assured. However if A is in the same direction as J then you probably can say this (before the addition of an arbitrary function with zero divergence). Also do note to be careful if you need to take derivatives of unit vectors in coordinates other than Cartesian!

Let’s try the same for a long wire of radius g which carries a uniform current density J. Inside the wire, we have from Ampere’s law:

$$\oint \vec{B}\cdot d\vec{L}=\mu_0 I=\mu_0 J\left(\pi s^2\right)\Rightarrow \vec{B}=\frac{1}{2}\mu_0 J s\hat{\phi}$$

Now we can calculate A inside the wire from this, but we will need to be careful about directions. From the useful page we have:

$$\vec{\nabla}\times\vec{V}=\left(\frac{1}{s}\frac{\partial V_z}{\partial\phi}-\frac{\partial V_\phi}{\partial z}\right)\hat{s}+\left(\frac{\partial V_s}{\partial z}-\frac{\partial V_z}{\partial s}\right)\hat{\phi}+\frac{1}{s}\left[\frac{\partial(sV_\phi)}{\partial s}-\frac{\partial V_s}{\partial\phi}\right]\hat{z}$$

We can thus formulate an A that will give B inside by looking at only the angular component:

$$\left(\frac{\partial V_s}{\partial z}-\frac{\partial V_z}{\partial s}\right)\hat{\phi}$$

Then let's suppose that A only has an s functional dependence in the curl. It can have other dependencies so long as all of those have zero curl.

$$\frac{\partial A_z}{\partial s}=-\frac{1}{2}\mu_0 J s\Rightarrow A_z=-\frac{\mu_0 J s^2}{4}+\text{constant}$$

Ok, then an A that will work is:

$$\vec{A}=-\frac{\mu_0 J s^2}{4}\hat{z}+\vec{\nabla} f(x,y,z)$$

where f is an arbitrary function with zero divergence. This is because the gradient of a divergence is zero. Notice that this is similar to the ambiguity in the electrostatic potential where we could have written that the potential had the addition of an arbitrary constant.

Let's confirm this gives B : $\vec{\nabla}\times\vec{A}=\frac{\mu_0 J}{4}\frac{\partial s^2}{\partial s}\hat{\phi}=\mu_0\frac{J}{2}s\hat{\phi}=B$.

Let's confirm this gives J: in Cartesian coordinates:

cylindrical coordinates,

$$\vec{\nabla}^2\vec{T}=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\vec{T}}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2\vec{T}}{\partial\phi^2}+\frac{\partial^2\vec{T}}{\partial z^2}$$

Here, A has only a z component and only an s functional dependence. So we have:

$$\vec{\nabla}^2\vec{A}=\frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\vec{A}}{\partial s}\right)=\frac{1}{s}\frac{\partial}{\partial s}\left(-\mu_0 J\frac{s^2}{2}\right)\hat{z}=-\mu_0 J\hat{z}=-\mu_0\vec{J}$$

If J has a radial dependence but still had a vector direction in z, I think you can see what would be necessary:

Suppose $\vec{J}=J_0\frac{s}{g}\hat{z}$

$$\oint\vec{B}\cdot d\vec{L}=\mu_0 I_c=\mu_0\oiint\vec{J}\cdot d\vec{A}\Rightarrow B(2\pi s)=\mu_0\frac{J_0}{g}2\pi\oiint s^2 ds=\mu_0\frac{J_0}{g}(2\pi)\frac{s^3}{3}$$

$$\Rightarrow B=\mu_0\frac{J_0}{g}\frac{s^2}{2}\hat{\phi}$$

Now let's get A:

$$A = \Phi_M = \oiint \vec{B}\cdot d\vec{a} = \mu_0 \frac{J_0}{g}\frac{1}{2}(2\pi)\frac{s^4}{4} = \mu_0 \frac{J_0}{g}\pi\frac{s^4}{4} \Rightarrow \vec{A} = \mu_0 \frac{J_0}{g}\pi\frac{s^4}{4}\hat{z}$$

The direction argument is the same here for A.

It would seem, then that an acceptable A can be calculated from B by determinations of the magnetic flux and then the direction of A can be argued from knowledge of J. However, since A is not uniquely defined, this result is not the only A under the sun that is correct. In current free regions, A is solved much in the same way as we solved for electrostatic potentials.

Let's see if we can set up one or more problems with the vector potential. Consider an infinite plane in the x-y plane carrying a surface current density

$$\vec{K} = |\vec{K}|\hat{x}$$

We know the magnetic field for this by Ampere's law:

$$\vec{B} = -\frac{\mu_0}{2}K\hat{y}(\text{if } z>0); \vec{B} = \frac{\mu_0}{2}K\hat{y}(\text{if } z<0)$$

The vector potential can not be obtained from: $\vec{A} = \frac{\mu_0}{4\pi}\oiint \frac{\vec{K}(\vec{r}_i)}{|\vec{r}_{ip}|}dA$ so easily because of divergences. However consider this:

$$\vec{B} = \vec{\nabla} x \vec{A}$$

Essentially, we require A to be a vector function whose curl only exists in the y direction and is independent of x,y and z. Then we have:

$$\vec{\nabla} x \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \hat{x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) - \hat{y}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right) + \hat{z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$(\vec{\nabla} x \vec{A})_y = -\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}$$

It is not reasonable to expect variation in A with respect to x it is also noted that B is in the y direction (ok, -y direction maybe). Thus I expect this to work: $\vec{A} = z(\text{constant})\hat{x}$ and this is reinforced by the current being in the x direction also. Remember, no matter how wrong you are with selections, in the end what ever A is eventually obtained, it needs to satisfy:

$$\nabla^2\vec{A} = -\mu_0\vec{J} : \vec{B} = \vec{\nabla} x \vec{A}$$

Which essentially is used to give us the directions for A from J.

For z>0, we then require: $\text{constant} = \frac{-\mu_0}{2}K$ and for z<0 we require: $\text{constant} = \frac{\mu_0}{2}K$ So an answer that will provide needed results is:

$$\vec{A} = -\frac{\mu_0}{2}Kz\hat{x}(z>0) : \vec{A} = \frac{\mu_0}{2}Kz\hat{x}(z<0)$$

Connection to reality time here: let's compare A at the point z=+w to A at the point z=-w: $\vec{A}_{+w}=-\frac{\mu_0}{2}Kw\,\hat{x}:\vec{A}_{-w}=\frac{\mu_0}{2}k(-w)\,\hat{x}$. So, with this A, the answers are the same above and below the plane. I think though both can be written into a single expression:

$$\vec{A}=-\frac{\mu_0}{2}K|z|\hat{x}$$

Notice that in order to use this you will need: $\frac{d|x|}{dx}=\frac{x}{|x|}$.

Notice that I have carefully avoided the region of the plane. In fact, the derivative above is not valid at x=0. In the regions provided, $\vec{\nabla}\,x\,\vec{A}=\vec{0}$ as required.

What about for a long wire (not here, necessarily infinite) carrying a current I running from -p to +p, I in the +z direction.

$$\vec{A}=\hat{z}\frac{\mu_0}{4\pi}I\int_{z=-p}^{z=+p}\frac{dz}{\sqrt{s^2+z^2}}=\hat{z}\frac{\mu_0}{4\pi}I\int_{z=-p}^{z=+p}\frac{dz}{\sqrt{s^2+z^2}}=\hat{z}\frac{\mu_0 I}{2\pi}\sinh^{-1}\left(\frac{p}{|s|}\right);s>0$$

This could also be written as:

$$\vec{A}=\hat{z}\frac{\mu_0 I}{2\pi}\ln\left(\frac{\sqrt{s^2+p^2}+p}{|s|}\right)$$

Now as p gets large this is approximately:

$$\vec{A}\approx\hat{z}\frac{\mu_0 I}{2\pi}\ln\left(\frac{2p}{s}\right)=\hat{z}(\text{constant})-\hat{z}\frac{\mu_0 I}{2\pi}\ln(s)$$

We are free to ignore the constant (infinite if a gets large) to obtain:

$$\vec{A}\approx-\hat{z}\frac{\mu_0}{2\pi}I\ln(s)$$

To get B, take the curl in cylindrical coordinates. Applying

$$\vec{\nabla}\times\vec{V}=\left(\frac{1}{s}\frac{\partial V_z}{\partial\phi}-\frac{\partial V_\phi}{\partial z}\right)\hat{s}+\left(\frac{\partial V_s}{\partial z}-\frac{\partial V_z}{\partial s}\right)\hat{\phi}+\frac{1}{s}\left[\frac{\partial(sV_\phi)}{\partial s}-\frac{\partial V_s}{\partial\phi}\right]\hat{z}$$

to our vector potential gives:

$$\vec{B}=\vec{\nabla}\,x\,\vec{A}=\hat{\phi}\left(--\frac{\mu_0 I}{2\pi}\left(\frac{1}{s}\right)\right)=\frac{\mu_0 I}{2\pi s}\hat{\phi}\quad .$$

Which is the result expected for an infinite wire. Also notice that it is less clean to do in the finite case but still doable.

Boundary conditions on B and A

From the problem involving the infinite sheet of current, we can find the boundary conditions when crossing a surface current density.

Here, a surface current is $\vec{K}=|\vec{K}|\hat{x}$. The infinite plane is in the x-y plane and z is the perpendicular distance away from the plane.for B parallel:

$$\vec{B}^{\parallel}_{above}=-\frac{\mu_0}{2}K\,\hat{y}:\vec{B}^{\parallel}_{below}=\frac{\mu_0}{2}K\,\hat{y}$$

Note the error in the diagram 5.50 on page 242.
This means there is a discontinuity in B parallel:

$$\triangle\vec{B}^{\parallel}\equiv\vec{B}^{\parallel}_{above}-\vec{B}^{\parallel}_{below}=-\mu_0 K\,\hat{y}$$

At least that is the sign for this particular problem as you move from -z to +z.

If, however, you just deal with magnitudes, this almost agrees with eq. 5.73 from your author. However I really think you need to be quite careful about vector directions when looking at these boundary conditions.

$$|\triangle\vec{B}^{\parallel}|\equiv|\vec{B}^{\parallel}_{above}-\vec{B}^{\parallel}_{below}|=\mu_0 K \quad .$$

Note that for both of these, the “parallel” refers to parallel to the surface which contains the surface current. This magnetic field, of course, is perpendicular to the direction of the surface current.

Now for the perpendicular direction of the current:let's look at a distant wire and call that a surface current. Then from Ampere's law, crossing the arbitrary plane of the wire shows easily that

$$\vec{B}^{\perp}_{above}=\vec{B}^{\perp}_{below}$$

Where the “perpendicular” here means perpendicular to both the current and the surface.
Both of these can be written as one expression:

$$\vec{B}_{above}-\vec{B}_{below}=\mu_0\vec{K}\,x\,\hat{n}$$

with $\hat{n}$ the normal vector to the surface defined by the surface currents and obtained in the usual way: $\hat{x}\,x\,\hat{y}=\hat{z}=\hat{n}$ for a surface in the x-y plane.

Now I have already shown that the vector potential is continuous when crossing the sheet of current (which is one of the big reasons I did it earlier).

$$\text{So:} \quad \vec{A}_{above} = \vec{A}_{below}$$

Note: this does not really have to be the case: in fact, you may have arbitrary constants added whose curl vanishes that are different above as compared to below. But it's better not to go there right now.

From the earlier problem, we had:

$$\vec{A} = -\frac{\mu_0}{2} K |z| \hat{x} \quad .$$

Let's take the normal derivative of this:

$$\text{Above the plane:} \quad \frac{\partial \vec{A}}{\partial n} = \frac{\partial \vec{A}}{\partial z} = \frac{-\mu_0}{2} K \frac{z}{|z|} \hat{x} = \frac{-\mu_0}{2} K \hat{x} (z > 0)$$

$$\text{Below the plane:} \quad \frac{\partial \vec{A}}{\partial n} = \frac{\partial \vec{A}}{\partial (-z)} = \frac{-\mu_0}{2} K \frac{z}{|z|} \hat{x} = \frac{\mu_0}{2} K \hat{x} (z < 0)$$

$$\text{So:} \quad \frac{\partial \vec{A}_{above}}{\partial n} - \frac{\partial \vec{A}_{below}}{\partial n} = -\mu_0 \vec{K}$$

which is in complete agreement with the author's result (eq. 5.76). I think it is important to be very careful to recognize n for z>0 is in the +z direction while n for z<0 is in the -z direction.

Multipole expansion of the vector potential

Remember for a charge distribution we developed:

$$\frac{1}{r_{ip}}=\frac{1}{r_p}\sum_{n=0}^{\infty}\left(\frac{r_{ip}}{r_p}\right)^n P_n(\cos(\theta_{ip}))$$

where it was required that we had azimuthal symmetry.
And we used this to develop a multipole expansion of the electrostatic potential from a charge distribution. We can do exactly the same thing here for a current distribution and the result is:

$$\vec{A}=\frac{\mu_0 I}{4\pi_p}\sum_{n=0}^{\infty}\frac{1}{r_p^{n+1}}\oint r_{ip}^n P_n(\cos(\theta_{ip}))\,d\vec{L}_i$$

In this expansion, the monopole term is always zero because the integration around a closed loop is zero. So we can actually start at n=1:

$$\vec{A}=\frac{\mu_0 I}{4\pi_p}\sum_{n=1}^{\infty}\frac{1}{r_p^{n+1}}\oint r_{ip}^n P_n(\cos(\theta_{ip}))\,d\vec{L}_i$$

Which also reflects no magnetic monopoles in nature. At least until they become observed in nature.

The dominate term then in the expansion is the dipole term (but apparently it can vanish also). When it does not vanish is what we are interested in here. If we just look at the dipole term, we have:

$$\vec{A}_{dip}=\frac{\mu_0 I}{4\pi}\frac{1}{r_p^2}\oint r_{ip}\cos(\theta_{ip})\,d\vec{L}_i$$

Now being sure to locate the dipole at the origin, we have:

$$\vec{A}_{dip}=\frac{\mu_0 I}{4\pi}\frac{1}{r_p^2}\oint(\hat{r}_p\cdot\vec{r}_{ip})\,d\vec{L}_i$$

In problem 1.61 (page 57), it is shown :

$$\oint(\hat{r}_p\cdot\vec{r}_{ip})\,d\vec{L}_i=-\hat{r}_p\times\oiint d\vec{a}_i$$

Then the dipole vector potential becomes:

$$\vec{A}_{dip}=\frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}_p}{r_p^2}$$

the **magnetic moment** defined by: $\vec{m}=I\oiint d\vec{a}=I\vec{a}$

The example of this: let m be at the origin, pointing in the z-direction. Then:

$$\vec{m}\times\hat{r}_p=m\hat{\phi}\Rightarrow\vec{A}_{dip}=\frac{\mu_0}{4\pi}\frac{m\sin(\theta)}{r_p^2}\hat{\phi}\quad\text{and}\quad\vec{B}=\vec{\nabla}\times\vec{A}=\frac{\mu_0 m}{4\pi r^3}(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})$$

Note: the coordinate-free version is: $\vec{B}=\frac{\mu_0}{4\pi}\frac{I}{r^3}[3(\vec{m}\cdot\hat{r})\hat{r}-m]$.