B from a spinning disk of charge 2016

A disk of radius a has a uniform surface charge density σ. It is spinning about its axis with a constant angular velocity ω. Calculate the magnetic field along the symmetry axis.

The law of Biot-Savert says

$$\vec{B}=\frac{\mu_0}{4\pi}\int_{\text{all currents}}\frac{\vec{K}\,x\,\vec{r}_{ip}}{\left|r_{ip}{}^{3}\right|}dA_i$$

where K is a surface current in the simplest case given by $\vec{K}=\sigma\vec{v}$

Here the problem is to find the current as a function of s. At a radius s on the disk, the velocity is given by $v=\omega s$. We can then obtain the surface current easily from this: $\vec{K}=\sigma\vec{v}=\sigma\omega s\hat{\phi}$. The vector can also be obtained from:

$$\vec{v}=\vec{\omega}\,x\,\vec{s}=\omega s[\hat{z}\,x(\cos\phi\,\hat{x}+\sin\phi\,\hat{y})]=\omega s\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\0&0&1\\\cos\phi&\sin\phi&0\end{vmatrix}=\omega s(-\sin\phi\,\hat{x}+\cos\phi\,\hat{y})=\omega s\hat{\phi}\quad .$$

Along the symmetry axis at a point z_p we have:

$$\vec{r}_i=x_i\hat{x}+y_i\hat{y}:\vec{r}_p=z_p\hat{z}:\vec{r}_{ip}=-s\cos\phi\,\hat{x}-s\sin\phi\,\hat{y}+z_p\hat{z}$$

if we put everything together, we then have:

$$\vec{B}=\frac{\mu_0}{4\pi}\oiint\frac{[(\sigma\omega s\hat{\phi})\,x\,(-s\cos\phi\,\hat{x}-s\sin\phi\,\hat{y}+z_p\hat{z})]}{\left[s^2+z_p^2\right]^{3/2}}s\,ds\,d\phi$$

$$\hat{\phi}\,x(s\hat{s}+z_p\hat{z})=\begin{vmatrix}\hat{x}&\hat{y}&\hat{z}\\-\sin\phi&\cos\phi&0\\s\cos\phi&s\sin\phi&z_p\end{vmatrix}=\hat{x}(z_p\cos\phi)-\hat{y}(-z_p\sin\phi)+\hat{z}(-s)$$

After integration through ϕ and by the right hand rule, along the symmetry axis only the z component will survive. The calculation for the magnetic field then reduces to:

$$\vec{B}=\frac{\mu_0}{4\pi}2\pi\sigma\omega\hat{z}\int_{s=0}^{s=a}\frac{s^3ds}{[s^2+z_p^2]^{3/2}}=\frac{1}{2}\mu_0\sigma\omega\hat{z}\int_{s=0}^{s=a}\frac{s^3ds}{[s^2+z_p^2]^{3/2}}$$

from the alpha site, int(x^3 /((x^2+ z^2)^(3/2))) from x=0 to x=a :

$$\int\frac{s^3ds}{[s^2+z_p^2]^{3/2}}=\frac{s^2+2z^2}{\sqrt{s^2+z^2}}$$

So: $\vec{B}=\frac{1}{2}\mu_0\sigma\omega\left[\frac{a^2+2z_p^2}{\sqrt{a^2+z_p^2}}-2|z_p|\right]\hat{z}$