B from a planar spiral 2016

planar spiral is in the x-y plane at z=0.

$$\vec{r}_i=a\phi\cos\phi\,\hat{x}+a\phi\sin\phi\,\hat{y}\quad .$$

starting at 2Mπ and ending at 2Nπ where a is a constant , not the radius.

We want to approximately calculate z along the symmetry axis.
The law of Biot-Savert says:

$$\vec{B}=\frac{\mu_0 I}{4\pi}\int\frac{d\vec{L}_i\,x\,\vec{r}_{ip}}{r_{ip}^3}$$

$$d\vec{r}_i=[a\cos\phi\,\hat{x}+a\sin\phi\,\hat{y}-a\phi\sin\phi\,\hat{x}+a\phi\cos\phi\,\hat{y}]\,d\phi$$

$$\vec{r}_{ip}=-a\phi\cos\phi\,\hat{x}-a\phi\sin\phi\,\hat{y}+z_p\hat{z}$$

$$d\vec{L}_i\,x\,\vec{r}_{ip}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ a(\cos\phi-\phi\sin\phi) & a(\sin\phi+\phi\cos\phi) & 0\\ -a\phi\cos\phi & -a\phi\sin\phi & z_p\end{vmatrix}d\phi$$

A simplified version of this is appropriate to start with. Let's just initially look at z=0:

$$d\vec{L}_i\,x\,\vec{r}_{ip}=\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ a(\cos\phi-\phi\sin\phi) & a(\sin\phi+\phi\cos\phi) & 0\\ -a\phi\cos\phi & -a\phi\sin\phi & 0\end{vmatrix}d\phi=$$

$$=\hat{z}\left(-a^2\phi\sin\phi(\cos\phi-\phi\sin\phi)+a^2\phi\cos\phi(\sin\phi+\phi\cos\phi)\right)d\phi=\hat{z}a^2\phi^2\,d\phi$$

The field integral is then given by: (note that we can not start at zero radius):

$$\vec{B}=\hat{z}a^2\frac{\mu_0 I}{4\pi}\int\frac{\phi^2 d\phi}{a^3\phi^3}=\hat{z}\frac{\mu_0 I}{4\pi}\frac{1}{a}\int_{2M\pi}^{2N\pi}\frac{d\phi}{\phi}=\hat{z}\frac{\mu_0 I}{4\pi a}\ln\left(\frac{N}{M}\right)$$

We need to work with a some since it is not exactly the radius. When the loop starts, the radius b is given by 2Mπa so $a=\frac{b}{2M\pi}$. Use this above: $\vec{B}=\frac{\mu_0 MI}{2b}\ln\left(\frac{N}{M}\right)\hat{z}$

Then suppose M=1 and N=2: $\vec{B}=\frac{\mu_0 I}{2b}\ln(2)\hat{z}$

Compare this to the result for a single loop at the center :

$$\vec{B}=\hat{z}\frac{\mu_0 NI}{2}\frac{a^2}{(a^2+z^2)^{3/2}}=\hat{z}\frac{\mu_0 I}{2a}$$

While the two results are similar, it needs to be pointed out that they are not exactly the same problem. However the first solution is expected to be smaller than the second and by maybe about 1/2 which is close to ln(2). A better comparison may come from looking at the magnetic field generated by a loop of radius a(1+π) or by averaging two fields, one from a loop of radius a and the other from a loop of radius a(1+2π).