B from a finite solenoid 2016

A solenoid is symmetrically placed with the end at z=0. It has a radius a and the coils are wound so that the equation which describes the winding is given by:

$$\vec{r}_i = a\cos\phi\,\hat{x} + a\sin\phi\,\hat{y} + c\phi\,\hat{z} \quad .$$

Here c is a (small) constant that can be negative or positive that is related to the winding separation.

We want to approximately calculate z along the symmetry axis.
The law of Biot-Savert says:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{d\vec{L}_i \, x\, \vec{r}_{ip}}{r_{ip}^3}$$

$$d\vec{r}_i = \left[-a\sin\phi\,\hat{x} + a\cos\phi\,\hat{y} + c\hat{z} \right] d\phi$$

$$\vec{r}_{ip} = -a\cos\phi\,\hat{x} - a\sin\phi\,\hat{y} + \left(z_p - c\phi\right)\hat{z}$$

$$d\vec{L}_i \, x\, \vec{r}_{ip} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -a\sin\phi & a\cos\phi & c \\ -a\cos\phi & -a\sin\phi & \left(z_p - c\phi\right) \end{vmatrix} d\phi =$$

$$\hat{x}\left(a\cos\phi\left(z_p - c\phi\right) + ac\sin\phi\right) d\phi - \hat{y}\left(-a\sin\phi\left(z_p - c\phi\right) + ac\cos\phi\right) d\phi + a^2\hat{z}\,d\phi$$

The magnetic field is then given by, assuming a total of N complete turns,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{\phi=0}^{2N\pi} \frac{\hat{x}\left(a\cos\phi\left(z - c\phi\right) + ac\sin\phi\right) - \hat{y}\left(-a\sin\phi\left(z - c\phi\right) + ac\cos\phi\right) + a^2\hat{z}}{\left(a^2 + \left(z - c\phi\right)\right)^{3/2}} d\phi$$

By symmetry, the cos and sin terms off to themselves will integrate away. We then have a simpler result:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{\phi=0}^{2N\pi} \frac{\hat{x}\left(a\cos\phi\left(z - c\phi\right)\right) + \hat{y}\left(a\sin\phi\left(z - c\phi\right)\right) + a^2\hat{z}}{\left(a^2 + \left(z - c\phi\right)\right)^{3/2}} d\phi$$

We can simplify further now using the same idea again:

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int_{\phi=0}^{2N\pi} \frac{-\hat{x}\left(ac\phi\cos\phi\right) - \hat{y}\left(ac\sin\phi\right) + a^2\hat{z}}{\left(a^2 + \left(z - c\phi\right)\right)^{3/2}} d\phi$$

Break this now into 3 integrals, doing the easiest and most important part first.

$$\vec{B}=\hat{z}\frac{\mu_0 I}{4\pi}\int_{\phi=0}^{2N\pi}\frac{a^2\,d\phi}{\left(a^2+\left(z-c\phi\right)^2\right)^{3/2}}=\hat{z}\frac{\mu_0 I}{4\pi}\left[\frac{c\phi-z}{c\sqrt{a^2+\left(z-c\phi\right)^2}}\right]_0^{2N\pi}=\hat{z}\frac{\mu_0 I}{4\pi}\left(\frac{2N\pi c-z}{c\sqrt{a^2+\left(z-2N\pi c\right)^2}}+\frac{z}{c\sqrt{a^2+z^2}}\right)$$

int a^2 / ((a ^2 + (z-c*x)^2)^(3/2))

Compare this to the ideal solenoid at the center. When traveling through a total distance of $d=2N\pi c$, 1/2 way through that is where $z=N\pi C$
At this point, we have:

$$\vec{B}=\hat{z}\frac{\mu_0 I}{4\pi}\left(\frac{N\pi c}{c\sqrt{a^2+\left(N\pi c\right)^2}}+\frac{N\pi c}{c\sqrt{a^2+\left(N\pi c\right)^2}}\right)=\hat{z}\frac{2\mu_0 I N}{4\sqrt{a^2+\left(N\pi c\right)^2}}$$

Assuming that a is fairly small, we then have:

$$\vec{B}\approx\frac{1}{2}\hat{z}\frac{\mu_0 I}{\pi c}$$

The total length of the solenoid is

$$2N\pi c=W\Rightarrow\pi c=\frac{W}{2N}\Rightarrow\vec{B}\approx\hat{z}\mu_0\frac{N}{W}I=\mu_0 n I\hat{z}$$

where n is the turn density. This is the result for the magnetic field near the center of the solenoid. Now let's get the factor of 1/2 for the field at the end of the solenoid.

Maybe the easiest way is to set z=0 to get:

$$\vec{B}=\hat{z}\frac{\mu_0 I}{4\pi}\left(\frac{2N\pi c}{c\sqrt{a^2+\left(2N\pi c\right)^2}}\right)$$

Again, assuming a is fairly small, we have:

$$\vec{B}=\frac{1}{2}\hat{z}\frac{\mu_0 I}{2\pi c}:2\pi c=\frac{W}{N}=\frac{1}{n}\Rightarrow\vec{B}=\frac{1}{2}\mu_0 n I\hat{z}$$

Which is the field at the end of a finite solenoid.

Notice that while the winding direction is controlled by the sign of c, if you start at z=0, with c negative, you will need to look at -z positions inside the solenoid. Probably it is worth looking at to get the winding direction.

We can get N planar loops from this same result: set N=1 and c=0. Then the x and y components exactly vanish and the integral for B reduces to:

$$\vec{B}=\hat{z}\frac{\mu_0 I}{4\pi}\int_{\phi=0}^{2N\pi}\frac{a^2\,d\phi}{\left(a^2+z^2\right)^{3/2}}=\hat{z}\frac{\mu_0 I}{4\pi}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}\left(2N\pi\right)=\hat{z}\frac{\mu_0 N I}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}$$