Spherical Coordinates3.3.2 2019

We are going to follow the lead of your author here and work only with problems that have azimuthal symmetry. If you need to break this symmetry, I'll refer you to "Classical Electromagnetic Radiation", Second Edition by Marion and Heald page 65 where the problem is given a more general treatment. The treatment that we'll give this problem will work in many situations and the more general solution is really not all that much more complicated. One note about the reference ... another set of electrostatic units is defined and used in that text.

If you refer to the useful page, you will find the Laplacian in spherical coordinates given as:

$$\vec{\nabla}^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial T}{\partial\theta}\right) + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 T}{\partial\phi^2}$$

In the present treatment, this reduces to:

$$\vec{\nabla}^2 T = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial T}{\partial\theta}\right)$$

Laplace's equation is:

$$\vec{\nabla}^2 V = 0$$

We thus need to solve:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial V}{\partial\theta}\right) = 0$$

We will separate this as before:

$$V = R\,\Theta$$

We thus have the simplified result:

$$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) + \frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\Theta}{\partial\theta}\right) = 0$$

Because people have worked with this before, we know how to most effectively choose the separation constant which is of the form l(l+1). We'll choose it so that:

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = l(l+1) : \frac{1}{\Theta}\frac{1}{\sin(\theta)}\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right) = -l(l+1)$$

Sorry that the fonts for 1 and l are so close.
We can write the solutions to the R equation:

$$R(r) = A\,r^l + \frac{B}{r^{l+1}}$$

The angular equation is more complicated.

$$\frac{d}{d\theta}\left(\sin(\theta)\frac{d\Theta}{d\theta}\right)=-l(l+1)\sin(\theta)\,\Theta$$

I want to write this in terms of x:

$$x\equiv\cos(\theta):\frac{d\Theta}{d\theta}=\frac{\partial\Theta}{\partial x}\frac{dx}{d\theta}=-\frac{d\Theta}{dx}\sin(\theta)$$

$$-\sin\theta\frac{d}{dx}\left(-\sin^2\theta\frac{d\Theta}{dx}\right)=\sin\theta\frac{d}{dx}\left((1-\cos^2\theta)\frac{d\Theta}{dx}\right)=\Rightarrow\frac{d}{dx}\left((x^2-1)\frac{d\Theta}{dx}\right)=l(l+1)\Theta$$

$$\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right)+l(l+1)\Theta=0$$

This is what will be the theta equation in terms of x.

These solutions are, however, well know to physicists … the solutions most useful are known as Legendre Polynomials:

$$\Theta(\theta)=P_l(\cos(\theta))$$

And the polynomials are most easily define by the Rodrigues formula:

$$P_l(x)=\frac{1}{2!l!}\left(\frac{d}{dx}\right)^l(x^2-1)^l$$

There is, of course, an entire other set of solutions but normally these are not used except for special geometries where, for example, the z-axis can not be reached, because of their divergent properties along the z-axis. The other orthogonal solution of Legendre's equation are $Q_l(x)$ which show singularities at x=+- 1 so we immediately discard these solutions. These are called Legendre functions of the second kind.

You can find out a lot of information about the Legendre polynomials at:
http://mathworld.wolfram.com/LegendrePolynomial.html
Here are the first few polynomials:

$$P_0(x)=1:P_1(x)=x\;:\;P_2(x)=\frac{1}{2}(3x^2-1)\;:\;P_3(x)=\frac{1}{2}(5x^3-3x)$$

$$P_4(x)=\frac{1}{8}(35x^4-30x^2+3)\;:\;P_5(x)=\frac{1}{8}(63x^5-70x^3+15x)$$

There are some important properties that you need to know about the Legendre Polynomials:

$$\int_{-1}^{+1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{mn}$$

You can evaluate other integrals involving Legendre Polynomials by reference to the URL given previously.

We thus can write the most general partial solution to V in this case:

$$V_l(r,\theta)=\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

The solution satisfying a particular set of boundary conditions will then be:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

I am going to work through the examples in your text.

Example 3.6

The potential is Vo(θ) on a sphere of radius R. Find the potential inside the sphere. The general solution is:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

We can immediately say that the B's are all zero since we don't want a divergence at the center of the sphere. Thus, the solution reduces to:

$$V(r,\theta)=\sum_{l=0}^{\infty}A_l r^l P_l(\cos(\theta))$$

On the surface of the sphere, we require:

$$V(R,\theta)=V_0(\theta)$$

We thus have:

$$V_0(\theta)=\sum_{l=0}^{\infty}A_l R^l P_l(\cos(\theta))$$

We now employ Fourier's trick:

$$\int_{\theta=0}^{\theta=\pi}V_0(\theta)P_{l'}(\cos(\theta))[\sin(\theta)]d\theta=\sum_{l=0}^{\infty}A_l R^l\int_{\theta=0}^{\theta=\pi}P_l(\cos(\theta))P_{l'}(\cos(\theta))[\sin(\theta)]d\theta$$

Look at the orthonormality condition above:

$$\int_{-1}^{+1}P_n(x)P_m(x)dx=\frac{2}{2n+1}\delta_{nm}$$

This can also be written as:

$$\int_{\theta=0}^{\theta=\pi}P_l(\cos(\theta))P_{l'}(\cos(\theta))\sin(\theta)d\theta=\frac{2}{2l'+1}\delta_{ll'}$$

(you'll notice a shift in the limits for this definition. It's ok here)

We thus see that the integral reduces to become:

$$\int_{\theta=0}^{\theta=\pi}V_0(\theta)P_{l'}(\cos(\theta))[\sin(\theta)]d\theta=\sum_{l=0}^{\infty}A_l R^l A_l R^l\frac{2}{2l'+1}\delta_{ll'}=\frac{2}{2l'+1}A_{l'}R^{l'}$$

We can now write the general solution for the A's:

$$A_l = \frac{2l+1}{2R^l} \int_{\theta=0}^{\theta=\pi} V_0(\theta) P_l(\cos(\theta)) [\sin(\theta)] d\theta$$

Your author refers to the "eyeball" method of solution for the A's. let's look at what he means.

Suppose that on the sphere, the potential is given by:

$$V_0(\theta) = k \sin^2\left(\frac{\theta}{2}\right)$$

Since $\sin^2\left(\frac{\theta}{2}\right) = \frac{1}{2}(1-\cos(\theta))$

we have:

$$V_0(\theta) = \frac{k}{2}(1-\cos(\theta)) = \frac{k}{2}\left(P_0(\cos(\theta)) - P_1(\cos(\theta))\right)$$

This means that we're only going to get two possibilities for the A's:

(1) L=0:

$$A_0 = \frac{k}{4R^0} \int_{\theta=0}^{\theta=\pi} P_0(\cos(\theta)) \sin(\theta) d\theta = \frac{k}{2R^0} = \frac{k}{2}$$

(2) L=1:

$$A_1 = -\frac{k}{2} \frac{2+1}{2R} \int_{\theta=0}^{\theta=\pi} P_1(\cos(\theta)) P_1(\cos(\theta)) \sin(\theta) d\theta = -\frac{k}{2} \frac{2+1}{2R} \frac{2}{2+1} = -\frac{k}{2R}$$

The solution to this problem is then:

$$V(r,\theta) = \sum_{l=0}^{1} A_l r^l P_l(\cos(\theta)) = \frac{k}{2} - k\frac{r}{2R}\cos(\theta) = \frac{k}{2}\left(1 - \frac{r}{R}\cos(\theta)\right)$$

We can write this more simply. Since $z = r\cos(\theta)$ **,**

we have:

$$V(r,\theta) = V(z) = \frac{k}{2}\left(1 - \frac{z}{R}\right)$$

which clearly shows a linear decrease in V as you go from z=-R to z=+R. I'm not sure why your author did not do this final step.

Let's now go ahead and solve for the region outside the sphere. In this case, we need a different set of solutions. We had in general:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

Here, however, the A's must go to zero and not the B's. Thus:

$$V(r,\theta)=\sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos(\theta))$$

Notice the typo in eq. 3.72 and in the equation following it.

As before, the angular dependence is going to be pretty straight-forward.

$$\frac{B_l}{R^{l+1}}=\begin{bmatrix} 0 & \text{if } l\neq 0 \text{ or } l\neq 1 \\ \frac{k}{2} & \text{if } l=0 \\ -\frac{k}{2} & \text{if } l=1 \end{bmatrix}\Rightarrow B_l=\begin{bmatrix} 0 & \text{if } l\neq 0 \text{ or } l\neq 1 \\ \frac{Rk}{2} & \text{if } l=0 \\ -R^2\frac{k}{2} & \text{if } l=1 \end{bmatrix}$$

The general solution is then:

$$V(r,\theta)=\frac{k}{2}\frac{R}{r}-\frac{k}{2}\frac{R^2}{r^2}P_1(\cos(\theta))=\frac{k}{2}\frac{R}{r}\left(1-\frac{R}{r}P_1(\cos(\theta))\right)$$

Let's now look at another application of this formalism.
Example 3.8

An uncharged conducting sphere of radius R is placed in an otherwise uniform electric field directed along the z axis. Find the potential in the region outside the sphere. You may assume the sphere is at zero potential.

In this case, the potential at infinity will diverge as $-E_0 z+C$. This kind-of means we're going to have to be careful what we throw away in the constants. We have as boundary conditions:

$$V=-E_0 z \text{ as } r\to\infty : V=0 \text{ for } r=R$$

The general solution from Laplace's equation is:

$$V(r,\theta)=\sum_{l=0}^{\infty}\left(A_l r^l+\frac{B_l}{r^{l+1}}\right)P_l(\cos(\theta))$$

At r=R, we require V=0. Thus, we have:

$$\left(A_l R^l+\frac{B_l}{R^{l+1}}\right)=0\Rightarrow B_l=-A_l R^{2l+1}$$

When r=R and θ=0, the potential must be equal to zero. This means that the L=0 term will not be present.
The boundary condition at infinity means that only the L=1 term will be present in the final solution. We thus have:

$$V(r,\theta)=A_1\left(r^1-\frac{R^3}{r^2}\right)\cos(\theta)$$

The final step is to fit A1: at infinity, we find:
$A1=-E_0$. Thus,

$$V(r,\theta)=-E_0\left(r^1-\frac{R^3}{r^2}\right)\cos(\theta)$$

You can easily find now the electric field:

$$\vec{E}=-\vec{\nabla}V=\frac{-\partial V}{\partial r}\hat{r}-\frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} \quad \textbf{Thus:} \quad \frac{\partial V}{\partial r}=-E_0\left(1+2\left(\frac{R}{r}\right)^3\right)\cos(\theta) : \frac{\partial V}{\partial\theta}=E_0\left(r-\frac{R^3}{r^2}\right)\sin(\theta)$$

we then have the electric field given by:

$$\vec{E}=E_0\left(1+2\left(\frac{R}{r}\right)^2\right)\cos(\theta)\hat{r}-E_0\left(1-\left(\frac{R}{r}\right)^3\right)\sin(\theta)\hat{\theta}$$

We can also find the induced surface charge density:

$$\sigma(\theta)=-\epsilon_0\left(\frac{\partial V}{\partial r}\right)_{r=R}=\left(\epsilon_0 E_0\left(1+2\left(\frac{R}{r}\right)^3\right)\cos(\theta)\right)_{r=R}=3\epsilon_0 E_0\cos(\theta)$$

Now it would not be correct to place two capacitor plates near the sphere (using this solution) and then try to determine a new potential on the capacitor plates. This is because we would be violating the assumed boundary condition that we used above.

We can, however, calculate the energy density of the field. This is given by:

$$u_E = \frac{1}{2}\epsilon_0 \vec{E}\cdot\vec{E}$$

Let's do this:

$$\vec{E} = E_0\left(1+2\left(\frac{R}{r}\right)^2\right)\cos(\theta)\hat{r} - E_0\left(1-\left(\frac{R}{r}\right)^3\right)\sin(\theta)\hat{\theta}$$

$$u_E = \frac{1}{2}\epsilon_0 E_0^2\left[\cos^2(\theta)\left(1+4\left(\frac{R}{r}\right)^2+4\left(\frac{R}{r}\right)^4\right)+\sin^2(\theta)\left(1-2\left(\frac{R}{r}\right)^3+\left(\frac{R}{r}\right)^6\right)\right]$$

$$u_E = \frac{1}{2}\epsilon_0 E_0^2\left[1+\cos^2(\theta)\left(4\left(\frac{R}{r}\right)^2+4\left(\frac{R}{r}\right)^4\right)+\sin^2(\theta)\left(-2\left(\frac{R}{r}\right)^3+\left(\frac{R}{r}\right)^6\right)\right]$$

This gives a function which is fairly smooth but does not particularly drop off to zero as you move away from the sphere. Don't try to integrate this over all space.