2.3.3 Laplace's Equation and Poisson's Equation

A quick review

For an electrostatic field:

$$\vec{E} = -\vec{\nabla} V$$

Also for the static E fields:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad \vec{\nabla} \times \vec{E} = \vec{0}$$

This then give Poisson's equation:

$$\vec{\nabla} \cdot \left(-\vec{\nabla} V\right) = \frac{\rho}{\epsilon_0} \Rightarrow \nabla^2 V = -\frac{\rho}{\epsilon_0}$$

In a charge-free region, this become Laplace's equation:

$$\nabla^2 V = 0$$

Which is most of free space.

We have seen the potential of a point charge and a charge distribution and know how to calculate it directly from integration.

We have also obtained the boundary conditions on E and V which are:

$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$

$$E^{\parallel}_{above} - E^{\parallel}_{below} = 0$$

$$V_{above} = V_{below}$$

$$\frac{\partial V_{above}}{\partial n} - \frac{\partial V_{below}}{\partial n} \equiv \vec{\nabla} V_{above} \cdot \hat{n} - \vec{\nabla} V_{below} \cdot \hat{n} = \frac{-\sigma}{\epsilon_0}$$

We have also calculate the work to assemble a charge distribution and the energy associate with a charge distribution.

2.5 Conductors (in electrostatic conditions)

Another quick review

Properties:

(i) $\vec{E}_{inside}=\vec{0}$

An external electric field produces induced charges in just the right way so that

$$\vec{E}_{inside}=\vec{0}\quad .$$

Namely the field from induced charges has got to be in the opposite direction.

(ii) Therefore $\rho_{inside}=0$

Because: $\vec{E}_{inside}=\vec{0};\ \frac{\rho}{\epsilon_0}=\vec{\nabla}\cdot\vec{E}\Rightarrow\rho=0$

(iii) Since conductors can carry a net charge, the charge must be on the surface and is expressed as a surface charge density (there is no where else for it to go).

(iv) A conductor is an equipotential surface (not, however, necessarily zero).

Why? $\Delta V=-\oint_a^b \vec{E}\cdot d\vec{L};\ \vec{E}_{inside}=\vec{0}\Rightarrow\Delta V=0$

(v) At the (external) surface of a conductor, E is normal. Why? because if there was a tangential component to E, charges would flow and make that tangential component vanish.

Holding a charge near a conductor will result in an attractive force because inside the conductor; a charge density will form in which the opposite charges are close to the external charge. This would not always be easy to calculate.

And finally a very clear statement about what if we hold a charge inside a simply connected void which is completely surrounded by conductor. The very clear statement is that there will be an electric field inside the void. However, the electric field will not penetrate the interior volume of the conductor. Never-the-less, a charge will form on an external surface of the conductor which has, as its origin, the internal charge held inside the void and thus; communicates the presence of the internal charge to the external world. This happens because of the requirement that the inner wall be equal and opposite to the charge held at the center: the charges are “pulled” from the surface during a short period of time which can not be characterized as “static”. I suggest close reading on pages 99, 100, and 101.

... which brings us to ....

2.5.3 Forces on a conductor

Inside a conductor the electric field is zero. From the boundary condition, we then have that right outside the conductor:

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{n}$$

If you have a surface charge present then the force is given by:

$$\vec{f} = \sigma \vec{E} \quad \text{(according to your author) (2.50)}$$

In fact, the corrected equation which makes sense is this:

$$\vec{f} = Q\vec{E} = \sigma A \vec{E}$$

In fact, because of the discontinuity in the electric field, we need to take an average:

$$\vec{f} = \frac{1}{2}\sigma A\left(\vec{E}_{above} + \vec{E}_{below}\right) = \sigma A \vec{E}_{average}$$

(read your author's words on page 103 about the patch)

In the particular case of the conductor, we have that the field inside is zero so

$$\vec{f} = \sigma A\left(\frac{\sigma}{2\epsilon_0}\right)\hat{n} = A\frac{\sigma^2}{2\epsilon_0}\hat{n}$$

(my corrected version of 2.51)

This force has the direction of pulling the surface into the direction of the electric field, independent of the sign of σ. It acts like an electrostatic pressure :

$$P \equiv \frac{f}{A} = \frac{A\sigma^2}{2\epsilon_0 A} = \frac{1}{2}\epsilon_0\left(\frac{\sigma}{\epsilon_0}\right) = \frac{1}{2}\epsilon_0 E^2 = u_E$$

(My version of 2.52)

Problem 2.37: Two large plates (each of area A) are held a distance d apart and each plate has a total charge Q (both positive in my version). What is the electrostatic pressure on the plates?

Between the plates, the electric field is zero. Outside the plates, the electric field is obtained by Gauss's law. Choose a cylinder with end area A' cutting through BOTH planes. Sketch the field lines.

$$\oiint \vec{E}\cdot dA = \frac{Q_{enc}}{\epsilon_0} \Rightarrow 2EA' = \frac{\sigma A'}{\epsilon_0} \Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

The pressure is then: $P = \frac{1}{2}\epsilon_0\left(\frac{\sigma}{2\epsilon_0}\right)^2 = \frac{1}{8}\frac{\sigma^2}{\epsilon_0}$

Which with a very simple argument is seen to be repulsive. If the plates have different charges, it is a different situation.

2.5.4 Capacitors

If we have conductors; place +Q on one and -Q on the other we calculate the potential difference as (from your author)

$$V = V_+ - V_- = -\oint_{(-)}^{(+)} \vec{E} \cdot d\vec{L}$$

If we knew the charge density, then:

$$\vec{E}_p = \int_{\text{all } q_i} k \frac{\rho(\vec{r}_i)}{r_{ip}^2} \hat{r}_{ip} d^3 r_i$$

The capacitance is define by:

$$C \equiv \frac{Q}{V}$$

and if we're calculating pure geometrical capacitance, the result should be independent of charge and it should be positive

Example which we'll experiment with:

The parallel plate capacitor:

At z=0, +σ. At z=d, -σ. Plates have an area A.
Find the electric field between the plates using Gauss's law (from phy250, for example):

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{z}$$

Calculate the potential difference:

$$V = -\frac{\sigma}{\epsilon_0} \int_{z=d}^{z=0} dz = -\frac{\sigma}{\epsilon_0}(0-d) = \frac{\sigma}{\epsilon_0} d$$

The total charge on the plates is given by:

$$Q = \sigma A$$

So, the capacitance is given by:

$$C = \frac{\sigma A}{\left(\frac{\sigma}{\epsilon_0} d\right)} = \epsilon_0 \frac{A}{d}$$

Which is the geometrical capacitance of the parallel plate capacitor.

Experiment Time with parallel plate capacitor.