2.1.4 Continuous charge distributions

We have talked about discrete charge distributions and obtained the result for the electric field:

$$\vec{E}_p = \sum_{i=1, i\neq p}^{n} k \frac{q_i}{r_{ip}^2} \hat{r}_{ip} \quad .$$

However, it is possible to also talk about continuous charge distributions. There are several simple geometries to cover here:

charge densities: linear: $dq_i = \lambda\, dx_i$: surface: $d\,q_i = \sigma\, dx_i dy_i$: volume: $dq_i = \rho\, dx_i dy_i dz_i$

With the use of the delta function, it is also possible to represent point charges.

For a linear charge distribution:

$$\vec{E}_p = \int_{\text{all } x_i} k \frac{\lambda(x_i) dx_i}{r_{ip}^2} \hat{r}_{ip}$$

For a surface charge distribution:

$$\vec{E}_p = \int_{\text{all } x_i, y_i} k \frac{\sigma(x_i, y_i) dx_i dy_i}{r_{ip}^2} \hat{r}_{ip}$$

For a volume charge distribution:

$$\vec{E}_p = \int_{\text{all } x_i, y_i, z_i} k \frac{\rho(x_i, y_i, z_i) dx_i dy_i dz_i}{r_{ip}^2} \hat{r}_{ip}$$

In each of the formulations, the integral is over the charges.

Example 2.1:

A wire has a length L and has a constant charge density λ. Find the electric field along the (positive) mid-sector. Note I have modified this problem slightly.

Let the wire lie along the x axis from x=-L/2 to x=+L/2. The total charge is given by:

$$Q = \int_{x=-L/2}^{x=+L/2} \lambda\, dx = \lambda [x]_{-L/2}^{+L/2} = \lambda L \Rightarrow \lambda = \frac{Q}{L}$$

Now find the various vectors we need:

$$p:(0, y_p, 0): i:(x_i, 0, 0): \vec{r}_{ip} = \vec{r}_p - \vec{r}_i = -x_i \hat{x} + y_p \hat{y}$$

Formulate the electric field:

$$\vec{E}_p = k \frac{Q}{L} \int_{x_i=-L/2}^{x_i=+L/2} \frac{(-x_i \hat{x} + y_p \hat{y})}{[x_i^2 + y_p^2]^{3/2}} dx_i$$

This integral is easy enough to do via that alpha site: integrate (x ^2 + y^2) ^(-3/2) dx

The solution for the y integral is $\displaystyle \int_{x_i=-L/2}^{x_i=+L/2} \frac{y_p dx_i}{[x_i^2 + y_p^2]^{3/2}} \left[\frac{y_p x_i}{y_p^2 \sqrt{x_i^2 + y_p^2}} \right]_{-L/2}^{+L/2} = \frac{L}{y_p \sqrt{(L/2)^2 + y_p^2}}$

The solution for the x integral is: $-\left[\frac{1}{\sqrt{x_i^2+y_p^2}}\right]_{x_i=-L/2}^{x_i=+L/2}=0$

We now put all this together:

$$\vec{E}_p=k\frac{Q}{L}\frac{L}{y_p\sqrt{L^2+y_p^2}}\hat{y}=\frac{kQ}{y_p\sqrt{(L/2)^2+y_p^2}}\hat{y}$$

We want to show though that for large y, the expected results obtain.

$$\vec{E}_p=\frac{kQ}{y_p^2\sqrt{\frac{L^2}{4y_p^2}+1}}\hat{y}\approx\frac{kQ}{y_p^2}\hat{y}=\frac{Q}{4\pi\epsilon_0 y_p^2}\hat{y}$$

However what about for large L?

$$\vec{E}_p=\frac{kQ}{(L/2)y_p\sqrt{1+\frac{4y_p^2}{L^2}}}\hat{y}\approx k\frac{2\lambda}{y_p}\hat{y}\Rightarrow\vec{E}_p\approx\frac{\lambda}{2\pi\epsilon_0 y_p}\hat{y}$$

This is the same result as will be obtained by application of Gauss's law to an infinitely long wire with a linear charge density λ.

I want to give two other examples now.

A plane with each side L has a uniform surface charge density σ. Find the Electric field in the region above the symmetry axis in this system. The plane is located in the x-y plane.

$$Q=\int_{y_i=-L/2}^{y_i=+L/2}\int_{x_i=-L/2}^{x_i=+L/2}\sigma\,dx_i\,dy_i=\sigma L^2\Rightarrow\sigma=\frac{Q}{L^2}$$

Calculate the vectors needed:

$$\vec{r}_i=x_i\hat{x}+y_i\hat{y}+0\hat{z}:\vec{r}_p=0\hat{x}+0\hat{y}+z_p\hat{z}:\vec{r}_{ip}=-x_i\hat{x}-y_i\hat{y}+z_p\hat{z}$$

Formulate E:

$$\vec{E}_p=k\frac{Q}{L^2}\int_{y_i=-\frac{L}{2}}^{y_i=+\frac{L}{2}}dy_i\int_{x_i=-\frac{L}{2}}^{x_i=+\frac{L}{2}}dx_i\frac{-x_i\hat{x}-y_i\hat{y}+z_p\hat{z}}{\left[x_i^2+y_i^2+z_p^2\right]^{3/2}}$$

The x and y components integrate to zero ... you can also use a vector argument here. So the only component which can survive is the z component.

Our electric field then reduces to:

$$\vec{E}_p=\hat{z}z_p k\frac{Q}{L^2}\int_{y_i=-\frac{L}{2}}^{y_i=+\frac{L}{2}}dy_i\int_{x_i=-\frac{L}{2}}^{x_i=+\frac{L}{2}}\frac{dx_i}{\left[x_i^2+y_i^2+z_p^2\right]^{3/2}}$$

At the alpha site: integrate (x ^2 + y ^2 + z ^2)^ (-3/2)

$$\int_{-L/2}^{L/2}dx\left(x^2+y^2+z^2\right)^{-3/2}=\left[\frac{x}{(y^2+z^2)\sqrt{x^2+y^2+z^2}}\right]_{-L/2}^{L/2}=\frac{L}{(y^2+z^2)\sqrt{(L/2)^2+y^2+z^2}}$$

And:

integrate (1/((x^2 + z ^2)*(a^2+x^2+z^2)^(1/2))

$$\int_{x=-L/2}^{L/2} \frac{dx}{\left(x^2+z^2\right)\sqrt{a^2+x^2+z^2}} = \left[\frac{\left[\tan^{-1}\left(\frac{ax}{z\sqrt{a^2+x^2+z^2}}\right)\right]}{az}\right]_{-L/2}^{L/2}$$

$$\int_{y=-L/2}^{L/2} \frac{dy}{\left(y^2+z^2\right)\sqrt{L^2/4+y^2+z^2}} = \frac{4}{Lz}\tan^{-1}\left[\frac{\frac{L^2}{4}}{z\sqrt{\frac{L^2}{2}+z^2}}\right]$$

So the electric field along the symmetry axis is given by:

$$\vec{E}_p = \hat{z}\, z_p k \frac{Q}{L^2} L \frac{4}{L z_p} \tan^{-1}\left[\frac{\frac{L^2}{4}}{z_p\sqrt{\frac{L^2}{2}+z_p^2}}\right]$$

We can easily examine this as z becomes large:

For large z we have:

$$\vec{E}_p \approx \hat{z}\, z_p k \frac{Q}{L^2} L \frac{4}{L z_p} \tan^{-1}\left[\frac{L}{2 z_p}\right]^2 \approx \hat{z}\, z_p k \frac{Q}{L^2} L \frac{4}{L z_p}\left(\frac{L}{2 z_p}\right)^2 = \hat{z}\, k Q \frac{1}{z_p^2} = k \frac{Q}{z_p^2}\hat{z}$$

What about as z gets small?

$$\vec{E}_p = \hat{z}\, z_p k \frac{Q}{L^2} L \frac{4}{L z_p} \tan^{-1}\left[\frac{1}{z_p}\right] \approx \hat{z}\, k 4 \sigma \frac{\pi}{2} = \hat{z}\frac{1}{4\pi\epsilon_0} 4\pi \frac{\sigma}{2} = \frac{\sigma}{2\epsilon_0}\hat{z}$$

Problem 2.6

As another example, consider a disk of radius a, charge Q in the x-y plane. Find the electric field above along the symmetry axis.

$$Q=\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi}\sigma r\,dr\,d\theta=\sigma\pi a^2\Rightarrow\sigma=\frac{Q}{\pi a^2}$$

$$\vec{r}_i=x_i\hat{x}+y_i\hat{y}+0\hat{z}:\vec{r}_p=0\hat{x}+0\hat{y}+z_p\hat{z}:\vec{r}_{ip}=-x_i\hat{x}-y_i\hat{y}+z_i\hat{z}$$

$$\vec{E}_p=k\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi}\frac{\sigma\left(-x_i\hat{x}-y_i\hat{y}+z_p\hat{z}\right)}{\left[x_i^2+y_i^2+z_p^2\right]^{(3/2)}}r\,dr\,d\theta$$

$$\vec{E}_p=k\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=2\pi}\frac{\sigma\left(-r\cos(\theta)\hat{x}-r\sin(\theta)\hat{y}+z_p\hat{z}\right)}{\left[r^2+z_p^2\right]^{(3/2)}}r\,dr\,d\theta$$

$$\vec{E}_p=2\pi k\int_{r=0}^{r=a}\frac{\sigma\left(z_p\hat{z}\right)}{\left[r^2+z_p^2\right]^{(3/2)}}r\,dr$$

$$\vec{E}_p=2\pi k\sigma z_p\hat{z}\int_{r=0}^{r=a}\frac{r\,dr}{\left[r^2+z_p^2\right]^{(3/2)}}$$

at the alpha site: int x /(x ^2 + z ^2) ^(3/2)

Result:

$$\vec{E}_p=2\pi k\sigma z_p\hat{z}\left(\frac{-1}{\sqrt{a^2+z_p^2}}+\frac{1}{|z_p|}\right)$$

For large z: we have:

$$\vec{E}_p=\frac{2\pi k\sigma z_p\hat{z}}{|z_p|}\left(\frac{-1}{\sqrt{\left(\frac{a}{z_p}\right)^2+1}}+1\right)=\frac{2\pi k\sigma z_p\hat{z}}{|z_p|}\left[-1+\frac{a^2}{2z_p^2}+1\right]=\frac{2\pi k\sigma z_p\hat{z}}{|z_p|}\frac{a^2}{2z_p^2}=\frac{Q}{4\pi\epsilon_0 z_p^2}\hat{z}$$

For small z: we have that as a gets large,

$$\vec{E}_p\approx 2\pi k\sigma z_p\hat{z}\left(\frac{1}{|z_p|}\right)=\frac{\sigma}{2\epsilon_0}\hat{z}$$

Note here is a useful reference for series expansions:
http://dlmf.nist.gov/4.6

A pipe of length L and radius q is located along the z-axis from -L/2 to +L/2. Find the electric field in the +z region if the pipe has a surface charge density.

$$Q=\int_{\theta=0}^{\theta=2\pi}\int_{z=-L/2}^{z=+L/2}\sigma a\,dz\,d\theta=\sigma(2\pi aL)\Rightarrow\sigma=\frac{Q}{2\pi aL}$$

$$\vec{r}_i=x_i\hat{x}+y_i\hat{y}+z_i\hat{z}:\vec{r}_p=0\hat{x}+0\hat{y}+z_p\hat{z}:\vec{r}_{ip}=-x_i\hat{x}-y_i\hat{y}+(z_p-z_i)\hat{z}$$

$$\vec{E}_p=k\sigma\int_{\theta=0}^{\theta=2\pi}\int_{z_i=-L/2}^{z_i=+L/2}\frac{-a\cos\theta\hat{x}-a\sin\theta\hat{y}+(z_p-z_i)\hat{z}}{\left[a^2+(z_p-z_i)^2\right]^{3/2}}d\theta\,dz_i$$

$$\vec{E}_p=\hat{z}2\pi k\sigma\int_{z_i=-L/2}^{z_i=+L/2}\frac{(z_p-z_i)dz_i}{\left[a^2+(z_p-z_i)^2\right]^{3/2}}$$

let $x\equiv z_p-z_i:dx=-dz_i:z_i=-L/2\Rightarrow x=z_p+L/2:z_i=+L/2\Rightarrow x=z_p-L/2$

$$\vec{E}_p=-\hat{z}2\pi k\sigma\int_{x=z_p+L/2}^{x=z_p-L/2}\frac{x\,dx}{\left[a^2+x^2\right]^{3/2}}$$

As before then:

$$\vec{E}_p=-\hat{z}2\pi k\sigma\left[\frac{-1}{\sqrt{a^2+x^2}}\right]_{z_p+L/2}^{z_p-L/2}=-\hat{z}2\pi k\sigma\left[\frac{1}{\sqrt{a^2+(z_p+L/2)^2}}-\frac{1}{\sqrt{a^2+(z_p-L/2)^2}}\right]$$