Curvilinear Coordinates

A: Unit Vectors

$$\hat{s}=\cos\phi\,\hat{x}+\sin\phi\,\hat{y}+0\,\hat{z}:\hat{\phi}=-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}+0\,\hat{z}:\hat{z}=0\,\hat{x}+0\,\hat{y}+\hat{z}$$

Cylindrical: $\hat{s},\hat{\phi},\hat{z}$

The transformation is given by:

$$x=s\cos\phi:y=s\sin\phi:z=z$$

We want to generate the unit vectors. There are lots of ways to do this but I think this is the easiest: Consider a vector: $\vec{s}=x\,\hat{x}+y\,\hat{y}$

Use the transformation above:

$$\vec{s}=s\cos\phi\,\hat{x}+s\sin\phi\,\hat{y}:\hat{s}=\frac{\vec{s}}{|\vec{s}|}\Rightarrow\hat{s}=\cos\phi\,\hat{x}+\sin\phi\,\hat{y}$$

Now we will assume that the coordinate direction z is generated by the cross product so that:

$$\hat{s}\,x\,\hat{\phi}=\hat{z}$$

Let's calculate what is needed now.

$$\hat{s}\,x\,\hat{\phi}=\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ \cos\phi & \sin\phi & 0\\ \phi_x & \phi_y & 0\end{bmatrix}=\hat{x}(0)-\hat{y}(0)+\hat{z}(\phi_y\cos\phi-\phi_x\sin\phi)$$

We want the magnitude to be one, which means that we can choose:

$$\phi_y=\cos\phi:\phi_x=-\sin\phi:\hat{\phi}=-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}+0\,\hat{z}$$

Spherical: $\hat{r},\hat{\phi},\hat{\theta}$ (look at problem 1.37)

The transformation is given by:

$$x=r\cos\phi\sin\theta:y=r\sin\phi\sin\theta:z=r\cos\theta$$

We want to generate the unit vectors.

Consider a vector: $\vec{r}=x\,\hat{x}+y\,\hat{y}+z\,\hat{z}$

Use the transformation above:

$$\vec{r}=r\cos\phi\sin\theta\,\hat{x}+r\sin\phi\sin\theta\,\hat{y}+r\cos\theta\,\hat{z}\Rightarrow\hat{r}=\cos\phi\sin\theta\,\hat{x}+\sin\phi\sin\theta\,\hat{y}+\cos\theta\,\hat{z}$$

We already know from the cylindrical coordinates that

$$\hat{\phi}=-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}$$

To generate the other component, I think it is easiest to take the dot product:

$$\hat{\theta}\cdot\hat{\phi}=-\theta_x\sin\phi+\theta_y\cos\phi\Rightarrow\theta_x=K(\cos\phi):\theta_y=K(\sin\phi)$$

We can ultimately calculate directly the unit vector by:

$$\hat{\phi}\,x\,\hat{r}=\hat{\theta}\Rightarrow\hat{\theta}=\begin{bmatrix}\hat{x} & \hat{y} & \hat{z}\\ -\sin\phi & \cos\phi & 0\\ \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta\end{bmatrix}$$

$$\Rightarrow\hat{\theta}=\hat{x}(\cos\phi\cos\theta)-\hat{y}(-\sin\phi\cos\theta)+\hat{z}(-\sin^2\phi\sin\theta-\cos^2\phi\sin\theta)$$

$$\Rightarrow\hat{\theta}=\cos\theta\cos\phi\,\hat{x}+\cos\theta\sin\phi\,\hat{y}-\sin\theta\,\hat{z}$$

Of course, we knew from the statement of the problem that $\hat{r}\,x\,\hat{\theta}=\hat{\phi}$

I'm not sure exactly how to know that except by trial and error. (Take Cartesian components of each vector and cross to make sure you have a right handed coordinate system).

Your author makes a very good point also: do not combine the vectors of two different point. (A=+y, B=-y both have the same r vector but the sum is zero, not 2r). Do operations with Cartesian coordinates or be very very careful.

We now want to generate the **spherical** equivalent of $d\vec{L}=dx\hat{x}+dy\hat{y}+dz\hat{z}$.

Our components are:

$$d\vec{L}=f_1\hat{r}+f_2\hat{\theta}+f_3\hat{\phi}$$

The easiest of these to see is $f_1=dr$ which simply moves along the radial direction. The other two correspond to arc length.

$$f_2=r\,d\theta$$

The third function is also arc length but it is the projection of r onto the x-y plane rotated through an angle. Thus:

$$f_3=r\sin\theta\,d\phi$$

The vector is then:

$$d\vec{L}=dr\hat{r}+r\,d\theta\hat{\theta}+r\sin\theta\,d\phi\hat{\phi}$$

We are now in a position to generate the volume element in spherical coordinates. The magnitude will be:

$$d^3r=f_1f_2f_3=r^2\sin\theta\,dr\,d\theta\,d\phi$$

We now want to generate the **cylindrical** equivalent of $d\vec{L}=dx\hat{x}+dy\hat{y}+dz\hat{z}$.

Our components are:

$$d\vec{L}=g_1\hat{s}+g_2\hat{\phi}+g_3\hat{z}$$

The easier of these to see are $g_1=ds:g_3=dz$

g_2 is generated as an arc length:

$$g_3=s\,d\phi$$

So our arc length in cylindrical coordinates is:

$$d\vec{L}=ds\hat{s}+s\,d\phi\hat{\phi}+dz\hat{z}$$

And we can find the magnitude of the volume element is:

$$d^3r=g_1g_2g_3=s\,ds\,d\phi\,dz$$

Important notes: do not take any unit vectors except Cartesian unit vectors outside integral signs.

Be careful when taking derivatives!!!

For example, in spherical coordinates,

Note: $\hat{r}=\cos\phi\sin\theta\hat{x}+\sin\phi\sin\theta\hat{y}+\cos\theta\hat{z}$ $\hat{\theta}=\cos\theta\cos\phi\hat{x}+\cos\theta\sin\phi\hat{y}-\sin\theta\hat{z}$

$$\frac{\partial\hat{r}}{\partial\theta}=\cos\phi\cos\theta\hat{x}+\sin\phi\cos\theta\hat{y}-\sin\theta\hat{z}=\hat{\theta}$$

The point is you need to take derivatives of the unit vectors also.

For the **curvilinear differential operator derivations**, I refer you to appendix A where the divergence theorem and Stokes' theorem are used to obtain the divergence and the curl in curvilinear coordinates.