Electric and Magnetic Fields in Linear Materials (improved)

The fundamental quantities are defined by the (electric/magnetic) susceptibility

Electric polarization: $\vec{P}$ =dipole moment per unit volume.

Do note this definition carefully! If I forget, remind me to talk about this.

The effect of an electric polarization is to produce a bound charge density:

$$\rho=\rho_f+\rho_b : \vec{\nabla}\cdot\vec{P}=-\frac{\rho_b}{\epsilon_0}$$

since we did this sheet first, you do not yet know why this is true

In any event, the divergence of the **Total electric field** gives the **Total** charge density:

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0} \quad \text{but} \quad \rho=\rho_f+\rho_b$$

So:

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho}{\epsilon_0}=\frac{\rho_f}{\epsilon_0}+\frac{\rho_b}{\epsilon_0}=\frac{\rho_f}{\epsilon_0}-\vec{\nabla}\cdot\vec{P}\Rightarrow\vec{\nabla}\cdot\left(\epsilon_0\vec{E}+\vec{P}\right)=\rho_f$$

The electric displacement vector is thus that vector whose divergence, in the presence of anything, gives the free charge density:

$$\vec{D}\equiv\epsilon_0\vec{E}+\vec{P}$$

Gauss's law then reads:

$$\vec{\nabla}\cdot\vec{D}=\rho_f\Leftrightarrow\oiint\vec{D}\cdot d\vec{A}=Q_{f_{enclosed}}$$

Here is a simple example for the calculation of D (Example 4.4) A wire is surrounded by a non-conducting material of radius a. The wire carries a surface charge density λ. Find D and E.

$$\oiint\vec{D}\cdot d\vec{A}=D\left(2\pi sh\right)=\lambda h\Rightarrow\vec{D}=\frac{\lambda}{2\pi s}\hat{s}$$

This holds for s<a and also for s>a. We can only find E outside the insulating material since we do not know the polarization:

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}\Rightarrow\vec{E}=\frac{\vec{D}}{\epsilon_0}-\frac{\vec{P}}{\epsilon_0}$$

Outside the material, the polarization is zero. We thus have:

$$\vec{E}=\frac{\lambda}{2\pi\epsilon_0 s}\hat{s}$$

Now in a lot of materials, the polarization is proportional to the electric field:

$$\vec{P}=\epsilon_0\chi_e\vec{E}$$

Here χ_e is the electric susceptibility, related to the dielectric constant by $\epsilon\equiv\epsilon_0\left(1+\chi_e\right)$

If the insulation were linear then we can calculate the electric field inside:

$$\vec{D}=\epsilon_0\vec{E}+\vec{P}\Rightarrow\vec{E}=\frac{\vec{D}}{\epsilon_0}-\frac{\vec{P}}{\epsilon_0}=\frac{\vec{D}}{\epsilon_0}-\frac{\epsilon_0\chi_e\vec{E}}{\epsilon_0}\Rightarrow\vec{E}=\frac{\vec{D}/\epsilon_0}{\left(1+\chi_e\right)}=\frac{\lambda}{2\pi s\epsilon_0\left(1+\chi_e\right)}\hat{s}=\frac{\lambda}{2\pi\epsilon s}\hat{x}$$

Now to calculate the electric potential for such systems, it is important to note that you need to integrate E, not D. Let's look at another problem (example 4.5): A metal sphere of radius a is surrounded out to radius b by a linear dielectric material of dielectric permittivity ϵ. Find all relevant electric quantities.

According to the prescription:

$$\oiint \vec{D}\cdot d\vec{A} = Q_{f_{enc}} \Rightarrow \vec{D} = \frac{Q}{4\pi r^2}\hat{r}$$

Now let's find E:

$$\vec{E} = \frac{\vec{D}}{\epsilon_0} = \begin{bmatrix} \frac{Q}{4\pi\epsilon_0 r^2}\hat{r}; r > b \\ \frac{Q}{4\pi\epsilon r^2}\hat{r}; a < r < b \\ 0; r \leq a \end{bmatrix}$$

Now we can find the polarization and thus the bound charge density:

$$\vec{P} = \epsilon_0\chi_e\vec{E} \Rightarrow \vec{P} = \epsilon_0\chi_e\frac{Q}{4\pi\epsilon r^2}\hat{r}$$

The bound charge density is given by:

$$\rho_b = -\vec{\nabla}\cdot\vec{P}$$

On both surfaces:

$$-\vec{\nabla}\cdot\vec{P} = 0$$

We get the surface charge density from the electric field at the surface:

$$\sigma_b = [\vec{P}\cdot\hat{n}]_{surface} = \begin{bmatrix} \frac{Q}{4\pi\epsilon b^2}; r = b \\ \frac{-Q}{4\pi\epsilon a^2}; r = a \end{bmatrix}$$

To obtain the potential, you need to integrate E:

$$V = -\int_{\infty}^{0}\vec{E}\cdot d\vec{s} = -\int_{\infty}^{b}\frac{Q}{4\pi\epsilon_0 r^2}dr - \int_{b}^{a}\frac{Q}{4\pi\epsilon r^2}dr$$

The boundary conditions when crossing a from dielectric 1 to dielectric 2 are:

$$\epsilon_2\vec{E}_2^{\perp} - \epsilon_1\vec{E}_1^{\perp} = \sigma_f : V_2 = V_1$$

We will come back here after a magnetic diversion. However in case I forget, skip to the end of these notes and don't forget.

Magnetization: $\vec{M}$ =magnetic dipole moment per unit volume

Magnetic fields arise from currents. Not all currents are free, some are bound.

The currents thus arising are given by:

$$\vec{J}=\vec{J}_f+\vec{J}_b$$

We thus define the magnetic induction vector as:

$$\vec{H}=\frac{1}{\mu_0}\vec{B}-\vec{M}$$

The magnetic susceptibility is defined (for linear materials) through:

$$\vec{B}=\mu_0\left(\vec{H}+\vec{M}\right) \quad \text{and} \quad \vec{M}=\chi_m\vec{H} \quad \text{(linear materials)}$$

The magnetic permeability is then given by:

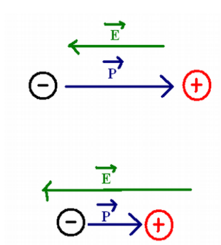
$$\mu\equiv\mu_0\left(1+\chi_m\right)$$

But you might ask yourself, now that you know about Maxwell's correction to Ampere's law, just where does $J_D=\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}$ fit into the picture? The answer is this: view B as arising from all currents (that's what currents do, they make B fields). Then the fields that arise specifically from free currents and are then given by:

note cross is control shift u 2a2f

$$\vec{\nabla}\times\left[\frac{1}{\mu_0}\vec{B}-\vec{M}\right]-\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}=\vec{J}_f\Rightarrow\vec{\nabla}\times\vec{H}-\epsilon_0\mu_0\frac{\partial\vec{E}}{\partial t}=\vec{J}_f$$

Now, let's consider depolarization currents. These currents will arise from a system depolarizing. I think a couple pictures will show what I want to here. First, how is the polarization vector defined for the electric dipole? The answer is that it points from the negative charge towards the positive charge in the physical dipole (see page 150, right below Eq. 3.101). Let's let polarization decrease in the system below.



Now looking at the change in polarization:

$\Delta\left|\vec{P}\right|<0;\Delta\vec{P}<0$ (the change in polarization is negative). What happens to the electric field at the center of the dipole though?

$$\Delta\left|\vec{E}\right|>0;\Delta\vec{E}<0$$

This means that the change in vector polarization was of the same sign as the change in electric field. If we suppose that there is a current which arises from this depolarization, then it would be expressed as:

$$\epsilon_0 \frac{\partial \vec{E}}{\partial t} = \vec{J}_p = \frac{\partial \vec{P}}{\partial t}$$

But notice please that the positive charge moving in the -x direction would imply a conventional current which was negative. I prefer to associate this just as I have shown because, unless we're talking about ionic conduction, depolarization would in my model come about by electric dipoles rotating to a more random position. I don't completely agree with your author's statement that this depolarization current is certainly not bound: consider for example atomic polarizability which is discussed on page 161. However, he is probably more right about this than I am.

Now look at how Maxwell placed his correction into Ampere's law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \vec{J}_D = \mu_0 \vec{J}_f + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

For currents arising from depolarization, we would include the term as

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \vec{J}_D = \mu_0 \vec{J}_f + \mu_0 \frac{\partial \vec{P}}{\partial t} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

I can write this in terms of D as:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \frac{\partial \vec{D}}{\partial t}$$

Now let's also let our system have some magnetization. This gives rise to a bound current through:

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

So the next step would be to include the contribution from these bound currents:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}_f + \mu_0 \vec{\nabla} \times \vec{M} + \mu_0 \frac{\partial \vec{D}}{\partial t}$$

I want to rewrite this equation now as:

$$\vec{\nabla} \times \left(\vec{B} - \mu_0 \vec{M}\right) = \mu_0 \vec{J}_f + \mu_0 \frac{\partial \vec{D}}{\partial t} \quad \text{or:}$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right) = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

But:

$$\vec{H} \equiv \left(\frac{1}{\mu_0} \vec{B} - \vec{M}\right)$$

so we would have in the presence of magnetic materials, the last of Maxwell's equations becomes:

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

The Magnetization is supposed to give rise to bound currents:

$$\vec{J}_b = \vec{\nabla} \times \vec{M}$$

Since both the free currents and the bound currents contribute to B, you really want some magnetic-field like quantity that arises solely from free currents. We can obtain this by subtracting off the part of the magnetic field that comes from magnetization and calculate it in the following way:

Since $\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$

It stands to reason that:

$$\vec{\nabla} \times \left[\frac{\vec{B}}{\mu_0} - \vec{M} \right] = \vec{J}_f$$

Now there is no reason that we would need to require $\vec{\nabla} \cdot \vec{M} = 0$: it might be nice and sometimes it may be true but in general, the best we can do for the second of Maxwell's equations is simply to say:

$$\vec{\nabla} \cdot \vec{B} = 0$$

Phenomenologically, it might be nice also if this were true:

$$\frac{\partial \vec{M}}{\partial t} = \vec{\nabla} \times \vec{E}$$

You don't see this type of term introduced in many texts ... but for sure it, or some other function of E depending upon magnetization changing does exist. In other words, it would be a nice symmetry if the depolarization producing a current were symmetric with the demagnetization producing charges. According to your author (see page 329) this is not the case. Well in a particular type of material called ferroelectric ferromagnets, it was indeed demonstrated that changing the direction of M does change the direction of E but it may not necessarily curl. There are not all that many examples of these types of materials out there but there are some.

This being the case, the third of Maxwell's equations is pretty much unchanged:

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

Ok, here are the four Maxwell equations for many materials:

$$\vec{\nabla}\cdot\vec{D}=\rho_f$$

$$\vec{\nabla}\cdot\vec{B}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{H}=\vec{J}_f+\frac{\partial\vec{D}}{\partial t}$$

and

$$\vec{D}\equiv\epsilon_0\vec{E}+\vec{P}\ \ :\ \ \vec{H}=\left(\frac{1}{\mu_0}\vec{B}-\vec{M}\right)$$

The boundary conditions on the various fields are:

$$\left[\epsilon_1\vec{E}_1-\epsilon_2\vec{E}_2\right]\cdot\hat{n}=\sigma_f:\left[\vec{B}_1-\vec{B}_2\right]\cdot\hat{n}=0$$

$$\left[\vec{E}_1-\vec{E}_2\right]\times\hat{n}=\vec{0}:\left[\frac{1}{\mu_1}\vec{B}_1-\frac{1}{\mu_2}\vec{B}_2\right]\times\hat{n}=\vec{K}_f\times\hat{n}$$

$\vec{K}_f$ is a free surface current density. It is usually assumed to be zero.

I think that it's conceptually easier to work with dielectric materials which are non-magnetic.

One interesting class of electric materials are ferroelectrics:

Ferroelectrics are a class of materials which retain their electric polarization. Consider just such a material which is, however, weakly ferroelectric.

$$\vec{P}=\vec{P}_{fe}$$

This system is certainly not linear. The ferroelectric polarization, up to the point that a large field is applied, is unmoved by the application of an external electric field.

You might wonder how to measure the polarization of such a system. The particular circuit used is straight-forward (so long as the systems are pretty good insulators). It is called the Sawyer Tower circuit (Phys Rev 35, 269 (1930)).

Here is also another, more modern reference

http://www.iue.tuwien.ac.at/phd/dragosits/node55.html