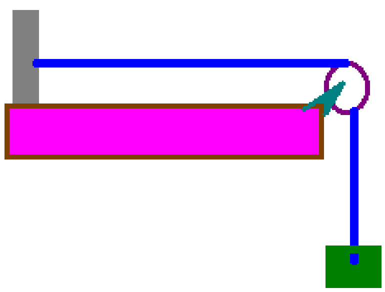
r21	Physics 240: Worksheet 23	Name: _______________

(1) A string has a mass of 1 kg and a length of 1000 m. The string is tied between two trees and has a tension of 100N. Find the frequencies of the modes of oscillation of the system.

(2) A string has a mass of 1 kg and a length of 10 m. The string is tied to a massless ring which can slide freely along a frictionless rod and the string is fixed to a tree on the other end. The string has a tension of 50N. Find the frequencies of the modes of oscillation of the system.

(3) A wire has a mass of 0.01 kg and a total length of 1.3 m. 0.3 m of the wire hangs over a pulley as shown and is attached to a 20 kg mass while the other end of the wire is attached to a grey block. Find the lowest 5 modes of oscillation in this system.



(4) Cowboy Ryan is continuing on his road trip and he decided to head out West! Right before passing through the mountains of western Wyoming, he decided to camp out in the grange overnight. During the night, with the campfire built, he pulled out his trusty 6 string guitar and thought of an interesting problem because he had been gone from my physics class for so long. Suppose that each of the strings on his guitar had the same mass, but the tension on each string was twice the tension on the previous string. Find the ratio of the lowest lying modes of oscillation on his guitar. If the lowest tension on the strings was 100N, and the length of each of the strings was 0.75m, and the mass of each string was 0.001kg, find the lowest frequencies of each of the strings.

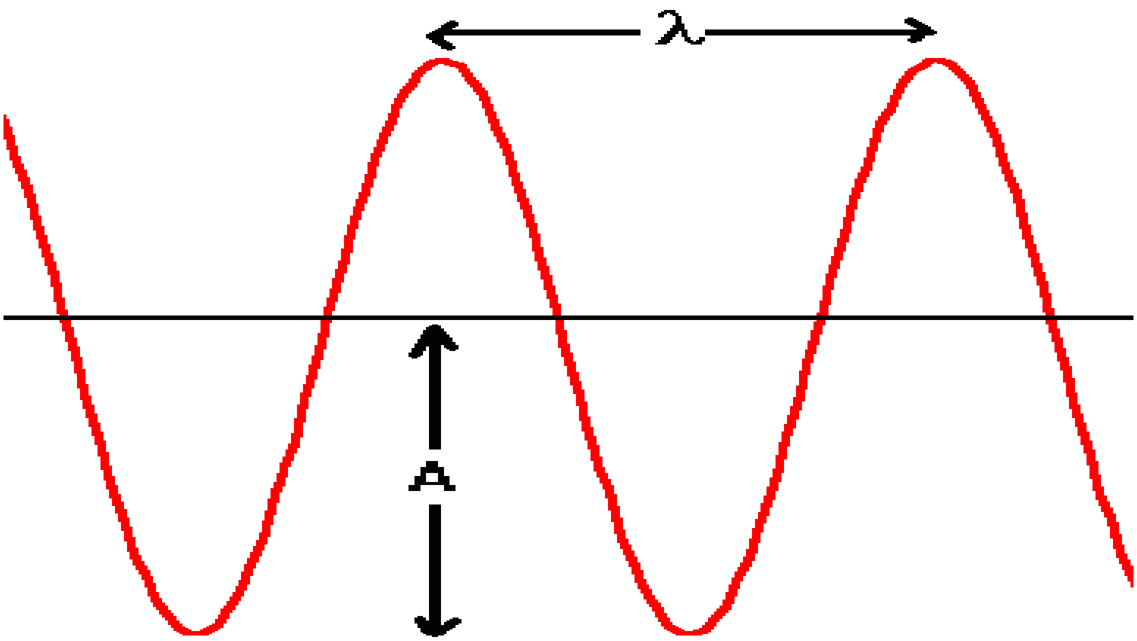
(5) An sinusoidal oscillator produces an amplitude of 0.05 m. The oscillator is attached to a string with a mass per unit length of 0.01 kg/m. How much power must the oscillator supply to the system if it is to operate at 60Hz under a tension of 50N?

Harmonic waves
A harmonic traveling wave is described by

$$y(x,t)=A\cos(kx\pm\omega t)$$

You may also see these waves described as:

$$y(x,t)=A\sin(kx\pm\omega t)$$



The wavelength of the wave is the length that the wave takes between identical amplitudes and velocities for a point as shown above. The amplitude is the maximum departure of a point from the equilibrium position.

How does this relate to our previous work with traveling pulses where we found that a pulse can be described by

$$Y(x,t)=f(x\pm vt)\ \ ?$$

Well suppose the particular function is this:

$$f(x\pm vt)=\sin(k(x\pm\frac{\omega}{k}t))$$

Then, in this context, it becomes clear that the argument implies the result that:

$$v=\frac{\omega}{k}$$

For a traveling wave of wavelength λ, we need to work out a relationship between velocity, wavelength and frequency. Let's see how this happens … this will help us get more understanding about the meaning of the *wave number* k.

Suppose that the traveling wave has a wavespeed v. Then in a time of 1 period, the wave will travel through a distance equal to the wavelength. This means,

$$vT=\lambda$$

We'll want to deal with frequency more than period when talking about waves, however. As you know, the two are easily related by

$$T=\frac{1}{f}$$

This then gives us the desired connection between the three quantities:

$$\left[vT=v\frac{1}{f}\right]=\lambda\Rightarrow\frac{v}{f}=\lambda\Rightarrow f\lambda=v$$

The last result is very important and you'll want to be sure you remember this.
Now, as promised, I want to show you more insight into k. Starting with:

$$f\lambda=v\Rightarrow(2\pi f)\left(\frac{\lambda}{2\pi}\right)=v$$

Ok, all I did here was multiply and divide by 2π.
Now, you recognize the angular frequency ω easily enough. Let's manipulate this a bit:

$$\omega\left(\frac{\lambda}{2\pi}\right)=v$$

We're almost there. You can compare this to the relation:

$$v=\frac{\omega}{k}$$

and you can see that if you make the identification

$$k\equiv\frac{2\pi}{\lambda}$$

Then the relationship clearly becomes the desired relationship!
This k is called the "wave number" although some people will call 1/λ the wavenumber. You'll need to know from text to text which is used. I'll use the first definition.

How do traveling harmonic waves transport energy?
Suppose a string has a mass per unit length μ.

A particular length (Δs) of the string then has a kinetic energy given by:

$$\Delta K=\frac{1}{2}mv_t^2=\frac{1}{2}(\mu\Delta s)v_t^2$$

Note that K is kinetic energy (not! wavenumber) and v_t is the transverse velocity of the small segment of string, not wavespeed.

Let's find out what the wavespeed is for a harmonic traveling wave.
Non-calculus students need only remember the result … it's not too easy to get this without calculus!

Our wave is described by:

$$y(x,t)=A\cos(kx\pm\omega t)$$

The transverse wave speed (transverse velocity) is defined by:

$$v_2=\frac{\partial y(x,t)}{\partial t}$$

which is the "partial derivative" of y. Now most of the calculus students may not have seen partial derivatives before. That's ok. The way you do these things is to take the derivative with respect to the particular variable treating everything that is not that variable as if it were a constant. In the present case, then we have:

$$\frac{\partial y(x,t)}{\partial t}=\mp\omega A\sin(kx\pm\omega t)$$

Thus, the instantaneous transverse velocity is given by:

$$v_t=\mp\omega A\sin(kx\pm\omega t)$$

Ok, so up to now, our kinetic energy of a small segment is given by:

$$\Delta K=\frac{1}{2}\mu(\Delta s)\omega^2A^2\sin^2(kx\pm\omega t)$$

The following argument follows like an IRS tax refund … as soon as they give it to you, you can be sure that they're going to take it back.

Now, it's not the instantaneous kinetic energy that we're really going to be interested in … we really want to know what the time average kinetic energy is. You know, however, how to find the time average of the $\sin^2$ function from our recent work on harmonic oscillators … think back … it is given by:

$$<\sin^2(kx-\omega t)>=\frac{1}{2}$$

so the time average kinetic energy is given by:

$$<\Delta K>=\frac{1}{2}\mu(\Delta s)\omega^2A^2\left(\frac{1}{2}\right)$$

Now, it turns out that if we only think about kinetic energy, we're really missing half of the picture because we also have ½ of the total energy stored in the string as elastic potential energy. Yes, I understand that I'm saying the string stretches and it doesn't stretch, but that's how this argument goes.

Thus, the total time average energy contained in a small segment of the string is given by:

$$<E>=2<K>=\frac{1}{2}\mu\omega^2 A^2(\Delta s)$$

Now, it's not really the time average energy that we want, we really want to know how fast this energy is being transported.
This is given by reference to the power (in SI units of watts, [W]):

$$P\equiv\frac{\Delta E}{\Delta t}$$

For our equation, then, we have:

$$P=\frac{<E>}{\Delta t}=\frac{1}{2}\mu\omega^2 A^2\left(\frac{\Delta s}{\Delta t}\right)$$

Now, you recognize the last term as the wave speed that we talked about earlier and also in the last lecture. Thus,

$$P=\frac{<E>}{\Delta t}=\frac{1}{2}\mu\omega^2 A^2 v$$

This is a very general result which applies somewhat to almost all transverse waves.

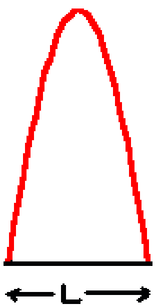
Standing waves on strings

The essential reason that standing waves occur on a string is because of a boundary. We have two boundary conditions to consider, fixed or free. As you'll see, both can give rise to standing waves.

Here is what happens (it's easiest to see for a fixed boundary condition) … a wave travels in and strikes the boundary. The reflected pulse goes back in the direction from which it came. If another wave pulse comes in at just the right time, parts of the wave will be constructively interfered with or destructively interfered with. The result then is that at the right frequency, you'll see a standing wave appear. I have produced an animated gif and also an excel spreadsheet that will let you see how this evolution takes place.

Modes of oscillation for fixed bc

Important here is that there is a node at each end of the string.

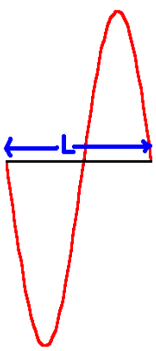


The lowest mode of oscillation (f_1)
½ of λ fits onto the length of the string. Thus:

$$\frac{\lambda_1}{2}=L\Rightarrow\lambda_1=2L\ .$$

The wavelength, speed and frequency are related by:
$f_1\lambda_1=v$. Here, v is the same for all frequencies since it is given by:
$v=\sqrt{\frac{T}{\mu}}$. We can then find the lowest frequency of oscillation by:

$$\left[f_1\lambda_1=f_1(2L)\right]=v\Rightarrow f_1=\frac{v}{2L}$$

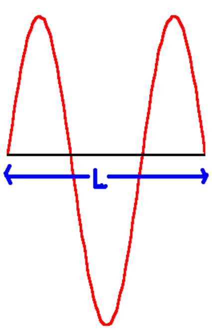


The second to lowest mode (f_2)
1 full λ fits onto the length of the string. Thus:

$$\lambda_2=L\ .$$

As before, we have:

$$\left[f_2\lambda_2=f_2L\right]=v\Rightarrow f_2=\frac{v}{L}=2\left(\frac{v}{2L}\right)=2f_1$$



The third to lowest mode (f_3)
1 ½ λ fits onto L. Thus:

$$\frac{3}{2}\lambda_3=L\Rightarrow\lambda_3=\frac{2L}{3}\ .$$

As before, we have:

$$\left[f_3\lambda_3=f_3\left(\frac{2L}{3}\right)\right]=v\Rightarrow f_3=\frac{3v}{2L}=3\left(\frac{v}{2L}\right)$$

It's pretty clear now that the n^{th} mode has a frequency given by:

$$f_n=n\left(\frac{v}{2}L\right)=nf_1,n=1,2,3,\ldots$$

Modes of oscillation for 1-free bc, 1 fixed. (mixed bc)

Important here is that there is a node at one end of the string and an antinode at the other.

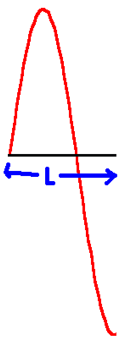
lowest mode of oscillation (f_1)

¼ of λ_1 fits onto L. Thus,

$$\frac{1}{4}\lambda_1 = L \Rightarrow \lambda_1 = 4L \quad .$$

The frequency, velocity and wavelength are related by:

$$\left[f_1 \lambda su1 = f_2 (4L)\right] = v \Rightarrow f_1 = \frac{v}{4L}$$



second to lowest mode (f_3)

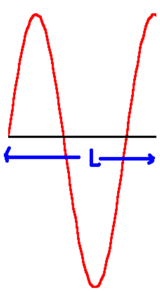
Note: I am calling this f_3 for reasons that will soon become clear.

¾ of λ_3 fits onto L. Thus,

$$\frac{3}{4}\lambda_3 = L \Rightarrow \lambda_3 = \frac{4}{3}L$$

As before:

$$\left[f_3 \lambda_3 = f_3\left(\frac{4}{3}L\right)\right] = v \Rightarrow f_3 = 3\left(\frac{v}{4L}\right) = 3f_1 \quad .$$



third to lowest mode (f_5)

Note: I am calling this f_5 for reasons that will soon become clear.

1 ¼ λ_5 fits onto L. Thus,

$$\frac{5}{4}\lambda_5 = L \Rightarrow \lambda_5 = \frac{4}{5}L$$

As before:

$$\left[f_5 \lambda_5 = f_5\left(\frac{4}{5}L\right)\right] = v \Rightarrow f_5 = 5\left(\frac{v}{4L}\right) = 5f_1$$

In general, it is now easy to see the result: $f_n = n\left(\frac{v}{4L}\right), n=1,3,5,\ldots$

$f_n = n(\frac{v}{4L}), n=1,3,5,\ldots$

We thus have the conclusion:

2 fixed bc $f_n = n\left(\frac{v}{2L}\right) = nf_1, n=1,2,3,\ldots$

1 fixed, 1 free bc $f_n = n\left(\frac{v}{4L}\right) = nf_1, n=1,3,5,\ldots$

Note that you can write the second condition as:

$$f_n=(2n-1)\left(\frac{v}{4L}\right), n=1,2,\ldots$$

Although, the actual meaning of n is slightly different between the two. Sometimes, however, it is more convenient to use the second expression.

(1) A string has a mass of 1 kg and a length of 1000 m. The string is tied between two trees and has a tension of 100N. Find the frequencies of the modes of oscillation of the system.
Solution:
for this system, we have 2 fixed bc. Thus, the modes of oscillation are given by:

$$f_n = n\left(\frac{v}{2}L\right) = n f_1, n=1,2,3,\ldots$$

We'll need to calculate v. This is given by:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{m/L}} = \sqrt{\frac{100}{1/1000}} = 316.2\,\frac{m}{s}$$

The frequency spectrum is then given by:

$$f_n = n\left(\frac{v}{2L}\right) = n\left(\frac{316.2}{2 \times 1000}\right) = n(0.158)\,Hz$$

The lowest modes are given by:

$$f_1 = 0.158\,Hz$$
$$f_2 = 0.316\,Hz$$
$$f_3 = 0.474\,Hz$$
$$f_4 = 0.632\,Hz$$
$$f_5 = 0.79\,Hz$$

(2) A string has a mass of 1 kg and a length of 10 m. The string is tied to a massless ring which can slide freely along a frictionless rod and the string is fixed to a tree on the other end. The string has a tension of 50N. Find the frequencies of the modes of oscillation of the system.

Solution:
for this system, we have mixed boundary conditions. Thus, the modes of oscillation are given by:

$$f_n = n\left(\frac{v}{4L}\right), n = 1,3,5,\ldots$$

We'll need to calculate v. This is given by:

$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{T}{m/L}} = \sqrt{\frac{50}{1/10}} = 22.4\,\frac{m}{s}$$

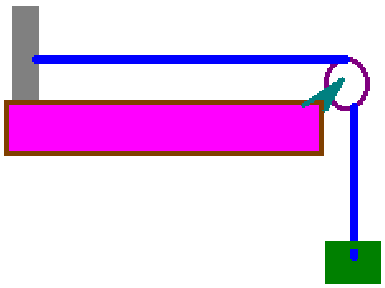
The frequency spectrum is then given by:

$$f_n = n\left(\frac{v}{4L}\right) = n\left(\frac{22.4}{4 \times 10}\right) = n(0.56)$$

The lowest modes are given by:

$$f_1 = 0.56\,\text{Hz}$$
$$f_3 = 1.68\,\text{Hz}$$
$$f_5 = 2.80\,\text{Hz}$$
$$f_7 = 3.92\,\text{Hz}$$
$$f_9 = 5.04\,\text{Hz}$$

(3) A wire has a mass of 0.01 kg and a total length of 1.3 m. 0.3 m of the wire hangs over a pulley as shown and is attached to a 20 kg mass while the other end of the wire is attached to a gray block. Find the lowest 5 modes of oscillation in this system.



Solution:
The tension in the wire is given by $T=[20+\mu(0.3)]g$ where I have not here ignored the small mass of the wire hanging over the pulley. The mass per unit length of the wire is given by $\mu=\frac{0.01\,kg}{1.3\,m}=7.69\,x\,10^{-3}\,kg/m$. Thus, the total tension on the wire is 196.02N. The small mass of the wire is not significant here.

The wave speed is given by

$$v=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{196.02\,N}{7.69\,x\,10^{-3}\,kg/m}}=159.7\,m/s$$

Now, here, it is important to realize that when calculating the frequencies of oscillation, the length to be used is 1.3 - .3 = 1.0m. I imagine the small length that wraps around the pulley is not significant.
This system most nearly approximates 2 fixed boundary conditions. Thus, the frequency spectrum is given by:

$$f_n=n\left(\frac{v}{2}L\right)=n\,f_1,\,n=1,2,3,\ldots$$

So, our frequencies are then:

$$f_n=n\left(\frac{159.7}{2\,x\,1}\right)=n(79.83)$$

The frequency spectrum for the 5 lowest lying modes is given by:

$$f_1=79.83\,Hz$$
$$f_2=159.7\,Hz$$
$$f_3=239.5\,Hz$$
$$f_4=319.3\,Hz$$
$$f_5=399.1\,Hz$$

(4) Cowboy Ryan is continuing on his road trip and he decided to head out West! Right before passing through the mountains of western Wyoming, he decided to camp out in the grange overnight. During the night, with the campfire built, he pulled out his trusty 6 string guitar and thought of an interesting problem because he had been gone from my physics class for so long. Suppose that each of the strings on his guitar had the same mass, but the tension on each string was twice the tension on the previous string. Find the ratio of the lowest lying modes of oscillation on his guitar. If the lowest tension on the strings was 100N, and the length of each of the strings was 0.75m, and the mass of each string was 0.001kg, find the lowest frequencies of each of the strings.

Solution:
The guitar approximates two fixed boundary conditions. Thus, the lowest lying frequency for each string is given by: $f_1 = \left(\frac{v}{2L}\right)$.

Each successive string doubles the tension. This means that the velocity of each successive string is going to increase. We can find the velocity by:

$$^{j}v = \sqrt{\frac{^{j}T}{\mu}}$$

where I've used the unusual left superscript j to designate a particular string. This means the frequency of the jth string will be given by:

$$^{j}f_1 = \frac{1}{2L}\sqrt{\frac{^{j}t}{\mu}} \Rightarrow \frac{^{j+1}f_1}{^{j}f_1} = \sqrt{\frac{^{j+1}T}{^{j}T}} = \sqrt{2} = 1.414$$

$$^{j}f_1 = \frac{1}{2L}\sqrt{\frac{^{j}T}{m}} \text{ Þ } \frac{^{j+1}f_1}{^{j}f_1} = \sqrt{\frac{^{j+1}T}{^{j}T}} = \sqrt{2} = 1.414$$

The frequencies are evenly separated by 1.414 Hz. Now, let's use the numbers to calculate the lowest frequency:

$$^{1}v = \sqrt{\frac{100\,N}{0.001\,kg/0.75\,m}} = 274\,\frac{m}{s} \Rightarrow {}^{1}f_1 = \frac{274}{2\,x.75} = 183\,Hz$$

The entire 6 lowest lying frequencies are then:

$$^{1}f_1 = 183\,Hz$$
$$^{2}f_1 = 258\,Hz$$
$$^{3}f_1 = 366\,Hz$$
$$^{5}f_1 = 518\,Hz$$
$$^{5}f_1 = 732\,Hz$$
$$^{6}f_1 = 1035\,Hz$$

(5) An sinusoidal oscillator produces an amplitude of 0.05 m. The oscillator is attached to a string with a mass per unit length of 0.01 kg/m. How much power must the oscillator supply to the system if it is to operate at 60Hz under a tension of 50N?
Solution:

The string carries off an amount of power given by:

$$P=\frac{<E>}{\Delta t}=\frac{1}{2}\mu\omega^2 A^2 v$$

at 60 Hz,

$$\omega=2\pi(60)=377\frac{rad}{s}$$

The wavespeed is given by

$$v=\sqrt{\frac{T}{\mu}}=\sqrt{\frac{50\,N}{0.01\,kg/m}}=70.7\frac{m}{s}$$

we can now calculate the power:

$$P=\frac{1}{2}\mu\omega^2 A^2 v=\frac{1}{2}(.01)(377)^2(0.05)^2(70)=124\,W$$

Notice that if you attach the oscillator to the end of a string, this is the correct answer. If, however, you attach the oscillator in the middle of the string, this is ½ of the correct answer since power is radiated into effectively two string segments.