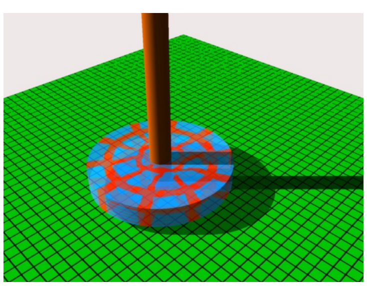
r20	Physics 240: Worksheet 20	Name: ________________

(1) A spring of spring constant k is attached to a mass m which is vibrating horizontally. Find the frequency of small oscillations about the equilibrium position.

(2) A mass (m=1 kg) is attached to a spring (k=50 N/m) and is stretched to an initial position x_0=0.3 m and released from rest. Find
(a) the frequency of oscillation
(b) the phase
(c) the amplitude
(d) the energy
(e) the position,
(f) the velocity,
(g) the acceleration

(3) A mass (m=5 kg) is attached to a spring (k=25 N/m) and is at an initial position x_0=0 and has an initial velocity v_0=+5 m/s.. Find:
(a) the frequency of oscillation
(b) the phase
(c) the amplitude
(d) the energy
(e) the position,
(f) the velocity,
(g) the acceleration

(4) A mass (m=50 kg) is attached to a spring (k=5 N/m) and is at an initial position x_0=+5 and has an initial velocity v_0=+5.5 m/s.. Find:
(a) the frequency of oscillation
(b) the phase
(c) the amplitude
(d) the energy
(e) the position,
(f) the velocity,
(g) the acceleration

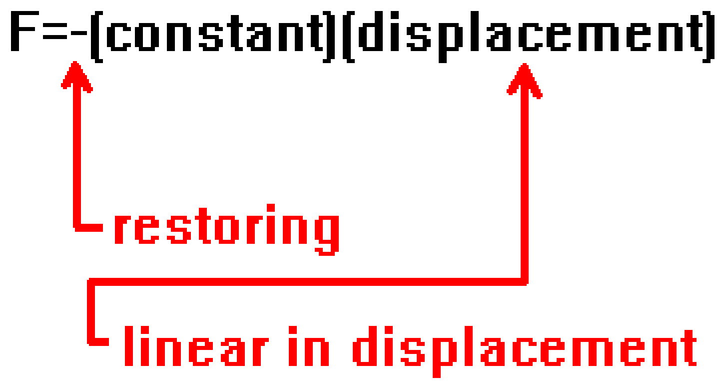


(5)A disk with a moment of inertia I is connected to a twistable rubber rod. The rod will apply a torque $\Gamma=-b\theta$ to the disk where b is a constant with units of Nm/"rad" and θ is the angle of twist. Find the frequency of small oscillations of the disk about equilibrium.

Simple Harmonic Oscillation

A simple harmonic oscillator is characterized by a system which has a ***linear*** (in the displacement variable) ***restoring*** force acting on a mass. The point here is that the form of the force is

$$F=-(\text{constant})x(\text{displacement})$$



Let's work out the spring-mass system as our first example.

(1) A spring of spring constant k is attached to a mass m which is vibrating horizontally. Find the frequency of small oscillations about the equilibrium position.

Solution: According to Hooke's law, $F=-kx$ where x is the displacement from equilibrium. Then, according to Newton's law:

$$F=ma=m\frac{dv}{dt}=m\frac{d^2x}{dt^2} \quad .$$

Let's equate these two expressions.

$$ma=-kx \quad \text{or}$$

$$m\frac{d^2x}{dt^2}+kx=0\Rightarrow\frac{d^2x}{dt^2}+\left(\frac{k}{m}\right)x=0$$

The acceleration of the system is given by

$$a=-\frac{k}{m}x$$

The motion about the equilibrium position is described by:

$$x: x(t)=A\cos(\omega t+\phi)$$

$$v: v(t)=\frac{dx}{dt}=-\omega A\sin(\omega t+\phi)$$

$$a: a(t)=\frac{dv}{dt}=-\omega^2 A\cos(\omega t+\phi)=-\omega^2 x$$

Note that you may also see the cosines written in terms of the sin function.

You can see that you have three undetermined constants, A, ω and ϕ. Let me show you by example how to model each of these constants. First, let's obtain ω. Look at the equation involving acceleration above:

$$a=-\frac{k}{m}x$$

Substitute in the expressions for x and a:

$$-\omega^2 x+\left(\frac{k}{m}\right)x=0\Rightarrow\omega=\sqrt{\frac{k}{m}}$$

The angular frequency comes out in a very straight-forward manner.
What about the other constants?

Suppose that at t=0 we know

$$x(t=0)=x_0 \text{ and } v(t=0)=v_0$$

Then, using these values for x(t) and v(t) above, we find:

$$x=A\cos(\phi) \text{ and } v=-\omega A\sin(\phi)\Rightarrow\frac{v_0}{x_0}=-\omega\tan(\phi)$$

$$\tan(\phi)=-\frac{v_0}{\omega x_0}$$

This in fact does not provide the complete story actually. Let's look more closely at the initial states: suppose $x_0=A\cos(\phi)$ is negative while $v_0=-\omega A\sin(\phi)$ is positive. Then you need to look a bit deeper: than the principle values. And also for the other way around. My advice is this: start with positive initial positions and positive initial velocities if possible. If one or the other is zero it is also easier. In any event, it is important to check that when you put the phase back into your equations of motion, you do get the expected initial values.

This provides us with the ***phase*** angle ϕ.

Now that we have ϕ, it is pretty simple to solve for A.

$$A=\frac{x_0}{\cos(\phi)}$$

A is called the ***"amplitude"*** and will give you the maximum displacement of the system from equilibrium.

What else can we find from this system: What about energy conservation? The total mechanical energy is given by:

$$E=K+U$$

$$K = \frac{1}{2} m v^2 = \frac{1}{2} m \left(-\omega A \sin(\omega t + \phi)\right)^2 = \frac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$$

$$U = \frac{1}{2} k x^2 = \frac{1}{2} k \left(A \cos(\omega t + \phi)\right)^2 = \frac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

We have already found that $\omega = \sqrt{\frac{k}{m}}$, Use this in the expression for K:

$$K = \frac{1}{2} k A^2 \sin^2(\omega t + \phi)$$

Now, add U to K to obtain:

$$E = \frac{1}{2} k A^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi)\right) = \frac{1}{2} k A^2 = \frac{1}{2} m \omega^2 A^2$$

The total energy of the harmonic oscillator is a constant of motion.

Here is a useful mathematical point ... the question is this ... What is the time average of something like $\sin^2(\omega t + \phi)$?.

Here is the world's simplest way to answer this question:
Consider the trig identity:

$$\cos^2(\theta)+\sin^2(\theta)=1$$

Here, θ might be allowed to change with time. Now, I'll use <func> to represent the "time average. Let's find the time average of these things:

$$<1>=1$$

(this follows from the observation that 1 always seems to be equal to 1)

$$<\sin^2(\omega t+\phi)>=<\cos^2(\omega t+\phi)>$$

(this follows from the observation that the two are the same curve but only shifted by a phase). Thus,

$$<\cos^2(\theta)>+<\sin^2(\theta)>=1 \Rightarrow \frac{1}{2}+\frac{1}{2}=1$$

or the time average of these trig functions is 1/2.

or ... $<\cos^2(\omega t+\phi)>=<\sin^2(\omega t+\phi)>=\frac{1}{2}$

Ok, now what is <U> and <K>?

$$<U>=<\frac{1}{2}kx^2>=\frac{1}{2}k<x^2>=\frac{1}{2}k<A^2\cos^2(\omega t+\phi)>=\frac{1}{2}kA^2\left(\frac{1}{2}\right)=\frac{1}{2}E$$

and

$$<K>=<\frac{1}{2}mv^2>=\frac{1}{2}m<\omega^2A^2\sin^2(\omega t+\phi)>=\frac{1}{2}m\omega^2A^2<\sin^2(\omega t+\phi)>=\frac{1}{2}m\omega^2A^2\left(\frac{1}{2}\right)=\frac{1}{2}E$$

Interestingly enough, the time average contributions from U and K are the same and each contributes on the average 1/2 of the total energy.

(2) A mass (m=1 kg) is attached to a spring (k=50 N/m) and is stretched to an initial position x_0=0.3 m and released from rest. Find
(a) the frequency of oscillation
(b) the phase
(c) the amplitude
(d) the energy
(e) the position,
(f) the velocity,
(g) the acceleration

(a) $\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{50}{1}}=7.07\frac{rad}{s}$

(b) $\tan(\phi)=\frac{-v_0}{\omega x_0}\Rightarrow\tan(\phi)=\frac{-0}{\omega x_0}=0\Rightarrow\phi=0$

Let's check this: at t=0, we have

$$x=.3\cos(0)=0.3 : v=-7.07\,x\,.3\,x\sin(0)=0$$

So the phase is correct. This pulls info from other parts of the problem.

(c) $x(t)=A\cos(\omega t+\phi)\Rightarrow x_0=A\cos(0)=A\Rightarrow A=x_0=0.3\,m$

(d) $E=\frac{1}{2}kA^2\Rightarrow E=\frac{1}{2}(50)(0.3)^2=2.25\,J$

(e) $x(t)=A\cos(\omega t+\phi)\Rightarrow x(t)=0.3\cos(7.07t)$

(f) $v(t)=-\omega A\sin(\omega t+\phi)\Rightarrow v(t)=-2.12\sin(7.07t)$

(g) $a(t)=-\omega^2 A\cos(\omega t+\phi)\Rightarrow a(t)=-15\cos(7.07t)$

(3) A mass (m=5 kg) is attached to a spring (k=25 N/m) and is at an initial position $x_0=0$ and has an initial velocity $v_0=+5$ m/s. Find:
(a) the frequency of oscillation
(b) the phase
(c) the amplitude
(d) the energy
(e) the position,
(f) the velocity,
(g) the acceleration

(a) $\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{25}{5}}=2.24\frac{rad}{s}$

(b) $\tan(\phi)=-\frac{v_0}{\omega x_0}=\frac{-5}{0}\Rightarrow\tan(\phi)=-\infty\Rightarrow\phi=-90^0$ or $-\frac{\pi}{2}$ radians

check: @t=0 : $x=A\cos(90)=0 \quad v_0=-2.24A\sin(-90)=2.24A$

(c) Here, use $v(t)=-\omega A\sin(\omega t+\phi)$ since $x_0=0$.

$$v_0(t)=-2.24A\sin\left(2.24\,x\,0-\frac{\pi}{2}\right)=-2.24A(-1)$$

$$\Rightarrow 2.24A=5\Rightarrow A=\frac{5}{2.24}=2.24m$$

We could get this from the total energy also:

$$E=\frac{1}{2}mv_0^2+\frac{1}{2}kx_0^2=\frac{1}{2}kA^2\Rightarrow A=\sqrt{\frac{m}{k}}v_0=\sqrt{\frac{5}{25}}(5)=2.24m$$

(d) at t=0, U=0 and $K=K_{max}$. Thus,

$$E=K_{max}=\frac{1}{2}mv_0^2=\frac{1}{2}(5)(5)^2=62.5J$$

(e) $x(t)=A\cos(\omega t+\phi)\Rightarrow x(t)=2.24\cos\left(2.24t-\frac{\pi}{2}\right)$

(f) $v(t)=-\omega A\sin(\omega t+\phi)\Rightarrow v(t)=-5\sin\left(2.24t-\frac{\pi}{2}\right)$

(g) $a(t)=-\omega^2A\cos(\omega t+\phi)\Rightarrow a(t)=-11.2\cos\left(2.24t-\frac{\pi}{2}\right)$

(4) A mass (m=50 kg) is attached to a spring (k=5 N/m) and is at an initial position x_0=+5 and has an initial velocity v_0=+5.5 m/s. Find:
(a) the frequency of oscillation
(b) the phase
(c) the amplitude
(d) the energy
(e) the position,
(f) the velocity,
(g) the acceleration

(a) $\omega=\sqrt{\frac{k}{m}}=\sqrt{\frac{5}{50}}=0.32\frac{rad}{s}$

(b) $\tan(\phi)=\frac{-v_0}{\omega x_0}\Rightarrow\tan(\phi)=\frac{-5.5}{.32 x 5}\Rightarrow\tan(\phi)=-3.44\Rightarrow\phi=-73.96^0$ or -1.29 radians

Check:

(c) We can use: $x(t)=A\cos(\omega t+\phi)$

$$x(0)=x_0=A\cos(.32 x 0-1.29)=0.276A\Rightarrow A=\frac{5}{0.276}=18.09m$$

We could get this from the total energy also:

$$E=\frac{1}{2}mv_0^2+\frac{1}{2}kx_0^2=\frac{1}{2}kA^2\Rightarrow\frac{1}{2}kA^2=\frac{1}{2}x50x5.5^2+\frac{1}{2}x5x5^2=756.25+62.5=818.75$$

$$A=\sqrt{2x818.75/5}=\sqrt{327.5}=18.1m$$

Make sure you were able to get the result shown above!
If you didn't, is your calculator set for the proper mode?

(d) $E=\frac{1}{2}kA^2=\frac{1}{2}(5)(18.1)^2=819J$

(e) $x(t)=A\cos(\omega t+\phi)\Rightarrow x(t)=18.1\cos(0.32t-1.29)$

(f) $v(t)=-\omega A\sin(\omega t+\phi)\Rightarrow v(t)=-5.792\sin(0.32t-1.29)$

(g) $a(t)=-\omega^2 A\cos(\omega t+\phi)\Rightarrow a(t)=-1.853\cos(0.32t-1.29)$

(5)A disk with a moment of inertia I is connected to a twistable rubber rod. The rod will apply a torque $\Gamma=-b\theta$ to the disk where b is a constant with units of Nm/"rad" and θ is the angle of twist. Find the frequency of small oscillations of the disk about equilibrium.
Solution:
This torque is a restoring torque: it always wants to restore the system to the angle θ=0. According to Newton's laws, an externally applied torque produces an angular acceleration:

$$\Gamma=I\alpha$$

We thus equate the two expressions to obtain:

$$I\alpha=-b\theta\Rightarrow\alpha=-\frac{b}{I}\theta$$

Calculus people recognize the differential equation:

$$\frac{d^2\theta}{dt^2}+\left(\frac{b}{I}\right)\theta=0$$

Compare this to the differential equation for the spring mass system, which was:

$$\frac{d^2x}{dt^2}+\left(\frac{k}{m}\right)x=0 \quad .$$

There, the frequency was $\omega=\sqrt{\frac{k}{m}}$ so by direct comparison, you can see that here the frequency of small oscillations is given by:

$$\omega=\sqrt{\frac{b}{I}}$$

but I can also show this a bit more directly:

Since: $\alpha=-\frac{b}{I}\theta$, consider a point on the disk at a distance r from the center. Multiply both sides of this equation by r to obtain:

$$r\alpha=-\frac{b}{I}r\theta\Rightarrow a=-\frac{b}{I}s$$

where a is the tangential acceleration and s is the position. From this description, it is then clear that the form is quite similar to the normal spring-mass system, and will rotate with the given frequency. Now suppose that the spring constant of the rubber rod was k and the disk has a mass m. Then there would also be a frequency of oscillation in the **vertical** direction. This frequency would be given by:

$$\omega=\sqrt{\frac{k}{m}} \quad .$$

If the two frequencies of oscillation were very close to each other, you would see an amazing result: the motion of the system would bounce completely with out twisting, then twist completely without bouncing and the two modes would be out of phase with each other if you started the system at the amplitude of one or the other. This is a form of resonance and the mechanism by which this happens is called “mode coupling” since the two modes are connected by a mechanical system.
This type of system is an example of a “Wilberforce pendulum” and you can see one in action here:

http://physics.kenyon.edu/coolphys/FranklinMiller/protected/wilber.html